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MATHEMATICS

Paper : 1.2

( **Topology** )

Full Marks : 80

Time : 3 hours

*The figures in the margin indicate full marks  
for the questions*

PART—A

( Objective-type Questions )

( Marks : 32 )

Each question (1-16) carries four responses (a), (b), (c) and (d) out of which one is for correct answer. Choose the correct response :  $2 \times 16 = 32$

1. A complete metric space is a metric space in which
- (a) every Cauchy sequence may not be convergent
  - (b) every metric is a discrete one
  - (c) every convergent sequence need not be a Cauchy sequence
  - (d) every Cauchy sequence is convergent

2. Let  $X$  be a complete metric space and let  $Y$  be a subspace of  $X$ . Then  $Y$  is complete if and only if
- (a) it is open
  - (b) it is closed
  - (c) it is a dense subset
  - (d) its interior is empty
3. Let  $X$  and  $Y$  be metric spaces and  $f$ , a mapping of  $X$  into  $Y$ . Then  $f$  is continuous if and only if
- (a)  $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$
  - (b)  $f(x_n) \rightarrow f(x) \Rightarrow x_n \rightarrow x$
  - (c) the inverse image of every open set in  $Y$  under  $f$  is closed in  $X$
  - (d)  $f$  is bijective
4. Consider the real line  $\mathbb{R}$  with the cofinite topology  $\tau$ . Select the collection which is not a base for  $\tau$
- (a) The collection of all finite subsets of  $\mathbb{R}$
  - (b) The collection consisting of the null set  $\phi$ , and all those subsets of  $\mathbb{R}$  whose complements are finite
  - (c) The collection consisting of the null set  $\phi$ ,  $\mathbb{R}$ , and all those subsets of  $\mathbb{R}$  whose complements are singleton sets
  - (d) The collection consisting of all those subsets of  $\mathbb{R}$  whose complements are  $\tau$ -closed

5. Let  $X$  be any non-empty set and let  $S$  be an arbitrary class of subsets of  $X$ . Then

- (a)  $S$  can always serve as a base for a unique topology on  $X$
- (b)  $S$  can always serve as a relative topology on any subset of  $X$
- (c)  $S$  can serve as a subbase for a unique topology on  $X$
- (d)  $S$  cannot serve as a subbase for any topology on  $X$

6. Select the false statement

- (a) Every closed and bounded subspace of the real line is compact
- (b) A continuous real or complex function defined on a compact space is bounded
- (c) Any continuous mapping of a compact metric space into a metric space is not uniformly continuous
- (d) A continuous real function  $f$  defined on a compact space  $X$  attains its infimum and its supremum

## 7. Select the true statement

- (a) The real line  $\mathbb{R}$  with the cofinite topology has the Bolzano-Weierstrass Property (BWP)
- (b) The real line  $\mathbb{R}$  with the usual topology has the BWP
- (c) The real line  $\mathbb{R}$  with the discrete topology has the BWP
- (d) The real line  $\mathbb{R}$  with the indiscrete topology cannot have the BWP

8. Consider the real line  $\mathbb{R}$  with the usual topology. Which one is a compact subset of  $\mathbb{R}$ ?

- (a) The open interval  $(-1, 1)$
- (b) The set of all positive real numbers
- (c) The union of all the intervals in the form  $(a, \infty)$ ,  $a \in \mathbb{R}$
- (d)  $[-1, 1] \cup [5, 8]$

## 9. Select the true statement

- (a) Every compact subspace of a Hausdorff space is open
- (b) A one-one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism
- (c) Every Hausdorff space is normal
- (d) Every regular space is a  $T_1$ -space

10. Consider the set of all natural numbers equipped with the topology

$$\tau = \{ \emptyset, \mathbb{N}, A_n = \{1, 2, 3, \dots, n\} \mid n \in \mathbb{N} \}$$

Select the true statement

- (a)  $\langle \mathbb{N}, \tau \rangle$  is a  $T_0$ -space
- (b)  $\langle \mathbb{N}, \tau \rangle$  is a  $T_2$ -space
- (c)  $\langle \mathbb{N}, \tau \rangle$  is normal, but not regular
- (d)  $\langle \mathbb{N}, \tau \rangle$  is both compact and connected
11. Let  $f$  be a continuous map from a compact space  $(X, \tau_1)$  onto a topological space  $(Y, \tau_2)$ . Select the true statement
- (a)  $(Y, \tau_2)$  is connected
- (b)  $(Y, \tau_2)$  is regular
- (c)  $(Y, \tau_2)$  is compact
- (d) None of the above
12. Consider the real line  $\mathbb{R}$  with the usual topology. Which one is a connected subset of  $\mathbb{R}$ ?
- (a)  $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$
- (b)  $(-1, 1)$
- (c)  $\mathbb{N}$
- (d)  $\mathbb{Q}$

13. Consider the onto map

$$f : [0, 1] \rightarrow [0, 1) \cup (5, 7]$$

Then  $f$  cannot be continuous, because

- (a)  $[0, 1]$  is not connected
  - (b)  $[0, 1) \cup (5, 7]$  is connected
  - (c)  $[0, 1) \cup (5, 7]$  is compact
  - (d)  $[0, 1) \cup (5, 7]$  is not compact
14. Which one of the following statements is false?
- (a) The components of a totally disconnected space are the null set  $\phi$  and the whole space
  - (b) Let  $X$  be a Hausdorff space. If  $X$  has an open base whose sets are also closed, then  $X$  is totally disconnected
  - (c) A topological space  $X$  is locally connected if the components of every open subspace of  $X$  are open in  $X$
  - (d) The range of a continuous real function defined on a connected space is an interval

15. Let

$$X = \{a, b, c\}$$

$$\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$$

Select the true statement

- (a) Every subset of  $X$  is both open and closed
- (b)  $(X, \tau)$  is a Hausdorff space
- (c)  $(X, \tau)$  is a regular space
- (d) None of the above

16. Select the true statement

- (a) Every base contains only open sets
- (b) Every base is a subbase
- (c) A topological space has only one base
- (d) A subbase always contains both open and closed sets

## PART—B

( Subjective-type Questions )

( Marks : 48 )

17. Answer any two parts : 6×2=12

(a) Let  $X$  be a complete metric space and let  $\{F_n\}$  be a decreasing sequence of non-empty closed subsets of  $X$  such that  $\delta(F_n) \rightarrow 0$ . Prove that  $F = \bigcap_{n=1}^{\infty} F_n$  contains exactly one point.

(b) If a complete metric space is the union of a sequence of its subsets, then show that the closure of at least one set in the sequence must have non-empty interior.

(c) Let  $X$  be a metric space, let  $Y$  be a complete metric space and let  $A$  be a dense subspace of  $X$ . If  $f$  is a uniformly continuous mapping of  $A$  into  $Y$ , then prove that  $f$  can be extended uniquely to a uniformly continuous mapping  $g$  of  $X$  into  $Y$ .

18. Answer any one part :

6

(a) Let

$$X = \{a, b, c\}$$

$$S = \{\{a, b\}, \{b, c\}\}$$

Obtain the topology generated by  $S$  as a subbase.

(b) Show that every separable metric space is second countable.

19. Answer any two parts :

$6 \times 2 = 12$

(a) Show that a continuous image of a compact space is compact.

(b) Prove that every closed and bounded subspace of the real line is compact.

(c) Establish the result :

"Every sequentially compact metric space is totally bounded."

20. Answer any two parts :

$6 \times 2 = 12$

(a) Prove that a topological space is a  $T_1$ -space  $\Leftrightarrow$  each finite set is a closed set.

(b) In a Hausdorff space, prove that any point and disjoint compact subspace can be separated by open sets.

- (c) Let  $X$  be a normal space and let  $A$  and  $B$  be disjoint closed subspaces of  $X$ . Prove that there exists a continuous real function  $f$  defined on  $X$ , all of whose values lie in the closed unit interval  $[0, 1]$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ .

21. Answer any one part :

6

- (a) Let  $X$  be a topological space. If  $\{A_i\}$  is a non-empty class of connected subspaces of  $X$  such that  $\bigcap A_i$  is non-empty, then show that  $A = \bigcup_i A_i$  is also a connected subspace of  $X$ .
- (b) If  $X$  is an arbitrary topological space, then prove the following :
- (i) Each point in  $X$  is contained in exactly one component of  $X$
  - (ii) Each connected subspace of  $X$  is contained in a component of  $X$

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