M 405

Institute of Distance and Open Learning
Gauhati University

M.A./M.Sc. in Mathematics
Semester 4

Paper V
General Theory of Relativity and Cosmology

Contents :

Math Paper V

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Unit I
Geodesics

1.0 Introduction :

A geodesics is the shortest arc between two points on a surface.It we stretch a rubber band between two points on a convex surface, the rubber band will take the path of the geodesic. A geodesic C on a surface S has the properties that at each ponts of C the principal normal coincides with the normal to S and the geodesic curvature vanishes indentically conversely, if on a curve C on a surface S, the principal normal coincides with the surface normal at every point, or if the geodesic curvature vanishes identically at every point, the curve is geodesic. If a straight line on a surface, then the line is a geodesic of the surface.

Althogh defining a geodesic as the shortest arc between two points on a surface gives the main idea of a geodesic, there is a problem with it as a definition. Not every geodesic is a shortest path in the large, as can be seen by noting that on the surface of a sphere every arc of a great circle is a geodesic even though on are will be the shrotest path between two points only if that arc is not greater than a semicircle. From this example we see that a geodesic can be a closed curve. Because of this difference a geodesic is often defined as an arc C on a surface S at each point of which the principal normal coincides with the normal to S or an are at every point of which the geodesic curvature vanishes identically.

1.1 Derivation of equation of geodesics :

A curve on a surface joining two points in an n-dimensional space is called the geodesic if its length is extremum.

The distance between any two points A and B lying on a surface is given by

$ds^2 = g_{ij}(x^i)\, dx^i\, dx^j$

considering the variation

$$\partial(ds^2) = \left(\frac{\partial g_{ij}}{\partial x^k}\partial x^k\right) dx^i dx^j + g_{ij}\partial(dx^i)dx^j + g_{ij}dx^i\partial(dx^j)$$

$$(i \leftrightarrow j)$$

$$\Rightarrow 2ds\,\partial(ds) = \frac{\partial g_{ij}}{\partial x^k}\partial x^k dx^i dx^j + 2g_{ij}\partial(dx^i)dx^j$$

$$\Rightarrow \partial(ds) = \left[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\partial x^k \frac{dx^i\,dx^j}{ds\,ds} + g_{ij}\frac{\partial(dx^i)}{ds}\frac{dx^j}{ds}\right]ds$$

$$\Rightarrow \int_A^B d(\partial s) = \int_A^B\left[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\partial x^k \frac{dx^i\,dx^j}{ds\,ds} + g_{ij}\frac{d}{ds}(\partial x^i)\frac{dx^j}{ds}\right]ds \qquad (1)$$

Now $\int_A^B\left(g_{ij}\frac{d}{ds}(\partial x^i)\frac{dx^j}{ds}\right)ds$

$$= \left[g_{ij}\frac{dx_j}{ds}\partial x^i\right]_A^B - \int_A^B \frac{d}{ds}\left(g_{ij}\frac{dx^j}{ds}\right)\partial x^i ds$$

$$= -\int_A^B \frac{d}{ds}\left(g_{ij}\frac{dx^j}{ds}\right)\partial x^i ds\left[\because \int \frac{d}{ds}\left(\partial x^i\right)ds = \partial x^i and \left[g_{ij}\frac{dx^j}{ds}\partial x^i\right]_A^B = 0\right].$$

from (i) we get

$$[\partial s]_A^B = \int_A^B\left[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds}\partial x^k - \frac{d}{ds}\left(g_{ij}\frac{dx^j}{ds}\right)\partial x^i\right]ds$$

$$(i \to k)$$

$$\Rightarrow 0 = \int_A^B\left[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds}\partial x^k - \frac{d}{ds}\left(g_{kj}\frac{dx^j}{ds}\right)\partial x^k\right]ds$$

$$\Rightarrow \int_A^B\left[\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{\partial g_{jk}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} - g_{jk}\frac{d^2x^j}{ds^2}\right]\partial x^k ds = 0$$

$$\left[\because \frac{d}{ds}\left(g_{kj}\right) = \frac{\partial g_{jk}}{\partial x^i}\frac{dx^i}{ds}\right]$$

Now for arbitrary ∂x^k ie $\partial x^k \neq 0$

We must have

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{\partial g_{jk}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} - g_{jk}\frac{d^2x^j}{ds^2} = 0$$

$$\Rightarrow \left(\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\right)\frac{dx^i}{ds}\frac{dx^j}{ds} - g_{jk}\frac{d^2x^j}{ds^2} = 0$$

$$(i \leftrightarrow j)$$

$$\Rightarrow\left(\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}-\frac{1}{2}\frac{\partial g_{jk}}{\partial x^i}-\frac{1}{2}\frac{\partial g_{ik}}{\partial x^j}\right)\frac{dx^i}{ds}\frac{dx^j}{ds}-g_{jk}\frac{d^2x^j}{ds^2}=0$$

$$\Rightarrow\frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\right]\frac{dx^i}{ds}\frac{dx^j}{ds}+g_{jk}\frac{d^2x^j}{ds^2}=0$$

$$\Rightarrow g^{\alpha k}g_{jk}\frac{d^2x^j}{ds^2}+g^{\alpha k}\Gamma_{ij,k}\frac{dx^i}{ds}\frac{dx^j}{ds}=0$$

$$\Rightarrow\delta^\alpha_j\frac{d^2x^j}{ds^2}+\Gamma^\alpha_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=o$$

$$\Rightarrow\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=0$$

which is the differential equation of geodesics.

Note :

The equation of geodesics in n dimensional space is given by

$$\Rightarrow\frac{d^2x^i}{ds^2}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0\left[i=1,2,...,n\right]$$

It consists of n-2nd order differential equations. But a 2nd order differential equation must contains two arbitrary constants in its general solutions.

Therefore the n-2nd order differential equation must involve 2n arbitary constants.

For the determination of these 2n arbitrary constants we must be given 2n condition. If the n co ordinates of a point $\left(x^1,x^2,x^3\,.........\,x^n\right)$ and the n components

$$t^j=\frac{dx^j}{ds}\left(t^i,t^2,.....t^n\right)$$

to characterise a direction of the geodesic through that point then the equation of geodesic can be uniquely determined.

Hence at every point through one derection, there exists one and only one direction.

1.2 Geodesic co ordinate system :

A co ordinate system x^i in an n dimensional space in called geodesic coordinate system with a pole P_0 it it is possible to construct a locally constant coordinate system at the pole P_0.

For geodesic coordinate system

$$\frac{dg_{ij}}{dx^k} = 0 \text{ at the pole } P_0$$

$$\neq 0 \text{ else where}$$

By definition of covariant derivative

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_\alpha \Gamma^\alpha_{ij}$$

But at P_0, $\Gamma_{ij,k} = 0 = \Gamma^i_{jk}$

Hence $A_{i,j} = \dfrac{\partial A_i}{\partial x^j}$ at P_0

Therefore at the pole P_0 geodesic coordinate system the covariant derivative of a vector or a tensor is equal to ordinary partial derivative.

1.3 Intrinsic derivative :

Consider a tensor $A^{ij\ldots k}_{ab\ldots c}$ defined along a curve $x^i = x^i(t)$ of parameter t. The intrinsic derivative is defined by the equation

$$\frac{\partial A^{ij\ldots\ldots k}_{ab\ldots\ldots c}}{\partial t} = A^{ij\ldots\ldots k}_{ab\ldots\ldots c},_p \frac{dx^p}{dt}$$

It means that the intrinsic derivative of a tensor is a tensor of the same and similar character as the original tensor.

Similarly intrinsic derivative of ϕ is

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x^i}\frac{dx^i}{dt} = \phi_{,i}\frac{dx^i}{dt}$$

. Also $\dfrac{\partial \phi}{\partial t} = \dfrac{d\phi}{dt}$

Thus intrinsic derivative of ϕ is its total derivative.

In brief the intrinsic derivative of a vector $\overline{u}(u^i)$ in the derection of a unit vector $\hat{a}$ is defined as

$u^i,_k\, a^k$

1.4 First curvature vector :

Let x^i be the co ordinate of any point on a curve C in an n dimensional space V_n. Let x^i expressed as the function of the parameter 's' so that

$$x^i = x^i(s)$$

The unit tangent vactor t^i to the curve C is defined as

$$t^i = \frac{dx^i}{ds}$$

1.5 Parallel displacement of a vactor :

A vector $\bar{u}$ of contravariant components u^i (or covariant components u_i) of constant magnitude is said to undergo a parallel displacement along a curve C if its intrinsic derivative along the direction of the curve vanishes

i.e $\quad u^i{}_{,k}\, a^k = 0$ or $u_{i,k} a^k = 0$

where a^k stands for components of the tangent vector to the curve C.

Solved problems

1. Find the differential equation of the geodesic in the space with the metric

$$ds^2 = -e^{2kt}\left(dx^2 + dy^2 + dz^2\right) + dt^2$$

Solution :

In this case

$$g_{11} = g_{22} = g_{33} - e^{2kt}$$
$$g_{44} = 1,\ g_{ij} = 0 \text{ when } i \neq j$$
$$x^1 = x,\ x^2 = y, x^3 = z, x^4 = t$$

The christoffel symbols are given by

$$\Gamma^1_{14} = k,\ \Gamma^2_{24} = k, \Gamma^3_{34} = k$$

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = ke^{2kt}$$

But the equation of geodesics are

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \qquad (1)$$

for $i=1$

$$\frac{d^2x^1}{ds^2}+\Gamma^1_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

or, $$\frac{d^2x}{ds^2}+\Gamma^1_{14}\frac{dx^1}{ds}\frac{dx^4}{ds}+\Gamma^1_{41}\frac{dx^4}{ds}\frac{dx^1}{ds}=0$$

or, $$\frac{d^2x}{ds^2}+2k\frac{dx}{ds}\frac{dt}{ds}=0 \qquad (2)$$

$$\left[\because \Gamma^1_{14}=\Gamma^1_{41}=k \text{ and } x^4=t\right]$$

for $i=2$

$$\frac{d^2x^2}{ds^2}+\Gamma^2_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

or, $$\frac{d^2y}{ds^2}+2k\frac{dy}{ds}\frac{dt}{ds}=0 \qquad (3)$$

for $i=3$ similarly we get

$$\frac{d^2z}{ds^2}+2k\frac{dz}{ds}\frac{dt}{ds}=0 \qquad (4)$$

But for $i=4$

$$\frac{d^2x^4}{ds^2}+\Gamma^4_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

or, $$\frac{d^2t}{ds^2}+k\,e^{2kt}\left[\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2\right]=0 \qquad (5)$$

Equation (2), (3), (4) and (5) are the differential equations of geodesies

From (2) we get

$$\frac{\frac{d}{ds}\left(\frac{dx}{ds}\right)}{\frac{dx}{ds}}=-2k\frac{dt}{ds}$$

i.e. $$\frac{d\left(\frac{dx}{ds}\right)}{\frac{dx}{ds}}=-2k\;dt$$

Integrating

$$\log\left(\frac{dx}{ds}\right) = -2kt + \log A$$

i.e. $$\log\left(\frac{1}{A}.\frac{dx}{ds}\right) = -2kt$$

i.e. $$\frac{dx}{ds} = Ae^{-2kt} \qquad (6)$$

Similarly,we can write the result of (3) and (4) as

$$\frac{dy}{ds} = Be^{-2kt} \qquad (7)$$

$$\frac{dz}{ds} = Ce^{-2kt} \qquad (8)$$

Putting the values of $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ in equation (5) we get

$$\frac{d^2t}{ds^2} + ke^{2kt}\left[\left(A^2 + B^2 + C^2\right)e^{-4kt}\right] = 0$$

or $$\frac{d^2t}{ds^2} + \alpha^2 e^{-2kt} = 0 \qquad (9)$$

where $\alpha^2 = k\left(A^2 + B^2 + C^2\right)$

From this equation 't' can be solved in terms of s (though non linear) which can be applied to (6), (7) and (8) for complete solution.

Equation (6), (7), (8) and (9) are actual differential equations of geodesic

2. The necessary and suffficient condition that a system of co ordinates be geodesic with pole at P_0 are that their second covariant derivatives w. r to the metric of the space all vanish at P_0.

Solution : We know that

$$\Gamma'^{p}_{ij} \frac{dx^l}{dx'^p} = \Gamma^l_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j} + \frac{d^2 x^l}{\partial x'^i \partial x'^j}$$

or $$-\Gamma^l_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j} = \frac{\partial}{\partial x'^j}\left(\frac{\partial x^l}{\partial x'^i}\right) - \Gamma'^{p}_{ij} \frac{\partial x^l}{\partial x'^p}$$

Interchanging x'^i and x^i system

$$-\Gamma'^{l}_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{i}}\frac{\partial x'^{\beta}}{\partial x^{j}}=\frac{\partial}{\partial x^{j}}\left(\frac{\partial x'^{l}}{\partial x^{i}}\right)-\Gamma^{p}_{ij}\frac{\partial x'^{l}}{\partial x^{p}} \qquad (1)$$

For a given value of l, x'^l is a scalar function of co-ordinates x^i so that

$$\frac{\partial x'^{l}}{\partial x^{i}}=x'^{l}_{,i}=x'^{l}_{i} \text{ (say)}$$

Now R.H.S of (1) is expressible as

$$\frac{\partial}{\partial x^{j}}\left(x'^{l}_{i}\right)-\Gamma^{p}_{ij}x'^{l}_{p}=x'^{l}_{i,j}=\left(x'^{l}_{i}\right)_{,j}=x'^{l}_{ij}$$

Using this in (1) we get

$$-\Gamma'^{l}_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{i}}\frac{\partial x'^{\beta}}{\partial x^{j}}=x'^{l}_{ij} \qquad (2)$$

Case (i) Let x'^l be a geodesic co-ordinate system with the pole at P_0 so that

$$-\Gamma'^{l}_{\alpha\beta}=0 \quad \text{at } P_0$$

Now (2) says that

$$x'^{l}_{ij}=0 \text{ at } P_0$$

Case (ii) Conversely suppose that

$$x'^{l}_{ij}=0 \text{ at } P_0 \qquad (3)$$

Then (2) gives

$$-\Gamma'^{l}_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{i}}\frac{\partial x'^{\beta}}{\partial x^{j}}=0 \text{ at } P_0$$

But $\qquad \dfrac{\partial x'^{\alpha}}{\partial x^{i}},\dfrac{\partial x'^{\beta}}{\partial x^{j}}\neq 0 \text{ at } P_0$

$$\therefore \Gamma'^{l}_{\alpha\beta}=0 \text{ at } P_0$$

This implies x'^l is a geodesic co ordinate system with the pole at P_0.

3. What is intrinsic derivative of a vector? If the intrinsic derivative of a vactor in any derection vanishes. Prove that it is of constant magnitude.

Solution :

The intrinsic derivative of a vactor $\overline{u}\left(u^{i}\right)$ in the directiove of a unit vector $\hat{a}$ is defined as $u^i_{,k}\, a^k$

Let the parametric representation of C be $x^i = x^i(s)$, s is the are length, then

$$a_{i,j}\frac{dx^j}{ds} = o \text{ along C} \qquad (1)$$

Let a be the magnitude of $\overline{a}$ this $a^2 = g_{ij}a^i a^j$

$$\frac{d}{ds}(a^2) = \frac{d}{dx^k}(a^2)\frac{dx^k}{ds}$$

$$\Rightarrow \frac{d}{ds}(a^2) = a^2_{,k}\frac{dx^k}{ds} = (g_{ij}a^i a^j)_{,k}\frac{dx^k}{ds}$$

$$= g_{ij}\left(a^i_{,k}a^j + a^i a^j_{,k}\right)\frac{dx^k}{ds}$$

$$= g_{ij}\left(a^i_{,k}\frac{dx^k}{ds}\right)a^j + g_{ij}a^i\left(a^j_{,k}\frac{dx^k}{ds}\right)$$

$= 0$ by (1)

$$\Rightarrow \frac{d}{ds}(a^2) = 0$$

$\Rightarrow a^2 =$ Constant

$\Rightarrow a =$ Constant

4. Define first curvature vactor p^i to be a curve C in V_n. What is the curve generated by $p^i = 0$ and what is its peculiarity?

Solution :

First part Ans see above

2nd part

The first curvature vector p^i to the curve C is defined by

$$p^i = t^i_{,k}\, t^k$$

Let us investigate the case when $p^i = 0$

$\therefore p^i = 0$ gives

$$t^i_{,k}\, t^k = 0$$

or, $\left(\frac{\partial t^i}{\partial x^k}+t^j\Gamma^i_{jk}\right)t^k=0\quad\left[\because t^k=\frac{dx^k}{ds}\right]$

or, $\frac{\partial t^i}{\partial x^k}\frac{dx^k}{ds}+t^jt^k\Gamma^i_{jk}=0$

or, $\frac{dt^i}{ds}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$

or $\frac{d}{ds}\left(\frac{dx^i}{ds}\right)+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$

or, $\frac{d^2x^i}{ds^2}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$

which is the equation of a very important curve, called the geodesic.

A geodesic is a curve of minimum arc length joining two points on a surface in any space.

5. If two vactors of constant magnitude undergo a parallel displacement along a curve C in V_n. Prove that they inclined at a constant angle.

Solution :

Let $\hat{a}(a^i)$ and $\hat{b}(b^j)$ be two unit vectors of constant magnitude

$\therefore$ If θ is the angle between them

$$cos\theta=g_{ij}a^ib^j=a_jb^j$$

$$\therefore\frac{d}{ds}(cos\theta)=\frac{d}{ds}\left(a_j\,b^j\right)$$

$$=\frac{d}{dx^k}\left(a_jb^j\right)\frac{dx^k}{ds}$$

$$=\left(a_jb^j\right)_{,k}\frac{dx^k}{ds}$$

$$=\left(a_{j,k}\,t^k\right)b^j+a_j\left(b^j_{,k}t^k\right)$$

$$=0$$

where t^k represents the direction of the curve C in V_n

$\therefore$ $\hat{a}$ and $\hat{b}$ suffer paralled displacement along C.

6. Obtain the non vanishing 3 index symbols for the metric

$$ds^2 = f(x)dx^2 + dy^2 + dz^2 + \frac{1}{f(x)}dt^2$$

and hence find the differential equations of the geodesics for this space.

Solution :

We have

$$ds^2 = f(x)dx^2 + dy^2 + dz^2 + \frac{1}{f(x)}dt^2$$

$$\therefore g_{11} = f(x), g_{22} = 1, g_{33} = 1, g_{44} = \frac{1}{f(x)}$$

and $g_{ij} = 0, i \neq j$

Also $x^1 = x, x^2 = y, x^3 = z, x^4 = t$

$\therefore$ for orthogonal space

$$g_{ij} = 0,\ i \neq j \text{ and } g^{ii} = \frac{1}{g^{ii}} \text{ for all } i$$

The non vanishing christoffel symbols are

$$\Gamma_{1,11} = \frac{1}{2}\frac{\partial f}{\partial x}, \Gamma_{4,41} = -\Gamma_{1,41} = \frac{-1}{2f^2}\frac{\partial f}{\partial x}$$

$$\Gamma^1_{11} = \frac{1}{2f}\frac{\partial f}{\partial x}, \Gamma^1_{44} = \frac{1}{2f^3}\frac{\partial f}{\partial x}\quad \Gamma^4_{41} = -\frac{1}{2f}\frac{\partial f}{\partial x}$$

$$\Gamma_{1,11} = \frac{1}{2}\frac{\partial f}{\partial x}, \Gamma_{4,41} = -\frac{1}{2f^2}\frac{\partial f}{\partial x}$$

$$\Gamma_{1,44} = \frac{1}{2f^2}\frac{\partial f}{\partial x}\quad \Gamma^1_{44} = \frac{1}{2f^3}\frac{\partial f}{\partial x}$$

The equation of geodesics are

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \qquad i = 1,2,3,4$$

for $i = 1$

$$\frac{d^2x}{ds^2} + \Gamma^1_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

or $$\frac{d^2x}{ds^2}+\Gamma^1_{11}\frac{dx^1}{ds}\frac{dx^1}{ds}+\Gamma^1_{44}\frac{dx^4}{ds}\frac{dx^4}{ds}=0$$

or $$\frac{d^2x}{ds^2}+\frac{1}{2f}\frac{\partial f}{\partial x}\left[\left(\frac{dx}{ds}\right)^2+\frac{1}{f^2}\left(\frac{dt}{ds}\right)^2\right]=0 \tag{1}$$

for $i=2$

$$\frac{d^2x^2}{ds^2}+\Gamma^2_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

or, $$\frac{d^2y}{ds^2}=0 \tag{2}$$

for $i=3$

$$\frac{d^2x^3}{ds^2}+\Gamma^3_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

or, $$\frac{d^2z}{ds^2}=0 \tag{3}$$

for $i=4$

$$\frac{d^2x^4}{ds^2}+\Gamma^4_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0$$

or, $$\frac{d^2t}{ds^2}+\Gamma^4_{41}\frac{dx^4}{ds}\frac{dx^1}{ds}+\Gamma^4_{14}\frac{dx^1}{ds}\frac{dx^4}{ds}=0$$

or, $$\frac{d^2t}{ds^2}-\frac{1}{f}\frac{\partial f}{\partial x}\frac{dx}{ds}\frac{dt}{ds}=0$$

or, $$\frac{\frac{d^2t}{ds^2}}{\frac{dt}{ds}}=\frac{1}{f}\frac{\partial f}{\partial x}\frac{dx}{ds}$$

$$=\frac{d}{ds}\left(\log f(x)\right)\frac{dx}{ds}$$

or, $$\int\frac{\frac{d^2t}{ds^2}}{\frac{dt}{ds}}ds=\int\frac{d}{ds}\left(\log f(x)\right)\frac{dx}{ds}ds+\log C$$

where log C is constant of integration

or, $$\log \frac{dt}{ds} = \log f(x) + \log C$$

or, $$\frac{dt}{ds} = Cf(x)$$

The equations (1), (2), (3) and (4) represents the required equations of geodesics

7. Suppose two unit vectors A^i, B^i are defined along a curve C such that their intrinsic derivative along C are zero. Show that angle between them is constant

Solution :

Let the vectors A^i and B^i be of constant magnitudes. Let these vectors suffer parallel displacement along a curve C, So that

$$\left.\begin{aligned} A^i_{,j}\frac{dx^j}{ds} = 0 \\ B^i_{,j}\frac{dx^j}{ds} = 0 \end{aligned}\right\} \qquad (1)$$

at each point of C

Let θ be the angle between A^i and B^i

To prove that θ = constant

$$A^i B_i = AB\ cos\theta$$

$$\frac{d}{ds}(AB\ cos\theta) = \frac{d}{ds}(A^i B_i)$$

$$= (A^i B_i)_{,i}\frac{dx^j}{ds}$$

or, $$-AB\,sin\theta\frac{d\theta}{ds} = A^i_{,j}\frac{dx^j}{ds}B_i + A^i B_{i,j}\frac{dx^j}{ds}$$

using (1) we get from (2)

$$-AB\,sin\theta\frac{d\theta}{ds} = 0$$

or, $$sin\theta\frac{d\theta}{ds} = 0$$

or, $\sin\theta = 0$ or $\frac{d\theta}{ds} = 0$

or $\theta = 0$ or, θ = constant.

or, θ = constant, for 0 is also a constant.

Exercise

1. Define Geodesics. Obtain the differential equations of geodesics in a given space
2. Obtain the non vanishing 3 index symbols for the metric

   $$dx^2 = -dx^2 - dy^2 - dz^2 + f(x,y,z)\,dt^2$$

   Hence obtain the equation of geodesics
3. If the intrinsic derivative of a vactopr vanishes then show that the magnitude of the vector is constant.

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Unit 2

Riemann Christoffel Curvature Tensors and Their Properties

2.0 Introduction :

In the mathematical field of differential geometry, the Riemann curvature tensor after Riemann and Christoffel is the most standard way to express curvature of Riemannian manifold. It associate a tensor to each point of a Riemannian manifold (i.e. it is a tensor field), that measures the extent to which the metric tensor is not locally isometric to Euclidian Space. The curvature tensor can also be defined for any pseudo-Riemannian manifold, or indeed any manifold equipped with an affine connection. It is a central mathematical tool in the theory of general relativity, the modern theory of gravity and the curvature of space-time is in principle observable via the geodesic deviation equation. The curvature tensor represents the tidal force experienced by a rigid body moving along a geodesic in a sense made precise by the Jacobi equation.

2.1 Riemannian Christoffel curvature tensor or curvature tensor :

We have seen that the covariant derivatives of three fundamental tensors g_{ij}, g^{ij} and g^i_j $(or\ \delta^i_j)$ are identically equal to zero. In other words there is no tensor of rank 3 which can be obtained entirely from fundamental tensors and their covariant derivatives. However there are two important tensors, one of rank two R_{ij} and the other of rank four R^{α}_{ijk} which involves only the first partial derivatives of the metric tensor and which are obtained from a repeated process of covariant differentiation.

Consider a covariant vector A_i, the covariant derivative of A_i w.r to x^j is given by

$$A_{i;j} = \frac{\partial A_i}{\partial x^j} - A_\alpha \Gamma^\alpha_{ij} \qquad (1)$$

which is a cavariant tensor of rank 2

A further covariant differentiation of equation (1) yields

$$\left(A_{i;j}\right)_{;k} \left(= A_{ij;k}\right) = \frac{\partial A_{ij}}{\partial x^k} - A_{\alpha j}\Gamma^\alpha_{ik} - A_{i\alpha}\Gamma^\alpha_{jk}$$

[writting after dropping comma i.e. $A_{i,j} = A_{ij}$]

$$= \frac{\partial A_{i,j}}{\partial x^k} - A_{\alpha,j}\Gamma^{\alpha}_{ik} - A_{i,\alpha}\Gamma^{\alpha}_{jk}$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j} - A_\alpha \Gamma^{\alpha}_{ij}\right) - \left(\frac{\partial A_\alpha}{\partial x^j} - A_\beta \Gamma^{\beta}_{\alpha j}\right)\Gamma^{\alpha}_{ik} - \left(\frac{\partial A_i}{\partial x_\alpha} - A_\beta \Gamma^{\beta}_{i\alpha}\right)\Gamma^{\alpha}_{jk}$$

$$= \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \frac{\partial A_\alpha}{\partial x^k}\Gamma^{\alpha}_{ij} - A_\alpha \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) - \frac{\partial A_\alpha}{\partial x^j}\Gamma^{\alpha}_{ik} + A_\beta \Gamma^{\beta}_{\alpha j}\Gamma^{\alpha}_{ik} - \frac{\partial A_i}{\partial x^\alpha}\Gamma^{\alpha}_{jk} + A_\beta \Gamma^{\beta}_{i\alpha}\Gamma^{\alpha}_{jk}$$

$$A_{ij:k} = \left[\frac{\partial^2 A_i}{\partial x^k \partial x^j} - \frac{\partial A_\alpha}{\partial x^k}\Gamma^{\alpha}_{ij} - \frac{\partial A_\alpha}{\partial x^j}\Gamma^{\alpha}_{ik} - \frac{\partial A_i}{\partial x^\alpha}\Gamma^{\alpha}_{jk} + A_\beta \Gamma^{\beta}_{i\alpha}\Gamma^{\alpha}_{jk}\right] - A_\alpha \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) + A_\beta \Gamma^{\beta}_{\alpha j}\Gamma^{\alpha}_{ik} \qquad (2)$$

Interchanging j and k in (2) we get

$$A_{ik:j} = \left[\frac{\partial^2 A_i}{\partial x^k \partial x^j} - \frac{\partial A_\alpha}{\partial x^j}\Gamma^{\alpha}_{ik} - \frac{\partial A_\alpha}{\partial x^k}\Gamma^{\alpha}_{ij} - \frac{\partial A_i}{\partial x^\alpha}\Gamma^{\alpha}_{kj} + A_\beta \Gamma^{\beta}_{i\alpha}\Gamma^{\alpha}_{kj}\right] - A_\alpha \frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ik}\right) + A_\beta \Gamma^{\beta}_{\alpha k}\Gamma^{\alpha}_{ij} \qquad (3)$$

$$(2) - (3) \Rightarrow A_{ij:k} - A_{ik:j} = A_\alpha\left[\frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) + \Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{ik} - \Gamma^{\alpha}_{\beta k}\Gamma^{\beta}_{ij}\right]$$

$$= A_\alpha R^{\alpha}{}_{ijk} \qquad (4)$$

where we have made of $R^{\alpha}{}_{ijk}$ in place of the bracket

i.e. $$R^{\alpha}{}_{ijk} = \frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) + \Gamma^{\alpha}_{\beta j}\Gamma^{\beta}_{ik} - \Gamma^{\alpha}_{\beta k}\Gamma^{\beta}_{ij} \qquad (5)$$

Since the L.H.S of equation (4) is the difference of two covariant tensors of real suffixes i,j,k therefore the R.H.S must also be a resulting tensor of same character. At the same time there is an arbitrary vector A_α on the R.H.S.

Hence the function representing the bracket must be of the form $R^{\alpha}{}_{ijk}$ so that the suffix α stand for a dummy suffix. In this case the R.H.S becomes the inner product of A_α with $R^{\alpha}{}_{ijk}$.

Hence the quotient law R^{α}_{ijk} must be a mixed tensor of the rank four. This is called the Riemann christoffel curvature tensor or simply curvature tensor of the 2nd kind.

Properties of $R^{\alpha}{}_{ijk}$

(i) $R^{\alpha}{}_{ijk} + R^{\alpha}{}_{jki} + R^{\alpha}{}_{kij} = 0$

Proof : By definition we have

$$R^{\alpha}{}_{ijk} = \frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) + \Gamma^{\alpha}_{bj}\Gamma^{b}_{ik} - \Gamma^{\alpha}_{bk}\Gamma^{b}_{ij} \tag{6}$$

$$R^{\alpha}_{jki} = \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ji}\right) - \frac{\partial}{\partial x^i}\left(\Gamma^{\alpha}_{jk}\right) + \Gamma^{\alpha}_{bk}\Gamma^{b}_{ji} - \Gamma^{\alpha}_{bi}\Gamma^{b}_{jk} \tag{7}$$

$$R^{\alpha}_{kij} = \frac{\partial}{\partial x^i}\left(\Gamma^{\alpha}_{kj}\right) - \frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ki}\right) + \Gamma^{\alpha}_{bi}\Gamma^{b}_{kj} - \Gamma^{\alpha}_{bj}\Gamma^{b}_{ki} \tag{8}$$

Adding (6), (7) and (8) we get

$$R^{\alpha}_{ijk} + R^{\alpha}_{jki} + R^{\alpha}_{kij} = 0$$

$$\because \Gamma^{\alpha}_{ij} = \Gamma^{\alpha}_{ji} \text{ etc.}$$

which is called the cyclic property of the curvature tensor.

(ii) $R^{\alpha}_{ijk} = -R^{\alpha}_{ikj}$

Proof : By definition

$$R^{\alpha}_{ijk} = \frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) + \Gamma^{\alpha}_{bj}\Gamma^{b}_{ik} - \Gamma^{\alpha}_{bk}\Gamma^{b}_{ij} \tag{9}$$

interchanging j and k

$$R^{\alpha}_{ikj} = \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) - \frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ik}\right) + \Gamma^{\alpha}_{bk}\Gamma^{b}_{ij} - \Gamma^{\alpha}_{bj}\Gamma^{b}_{ik}$$

$$= -\left[\frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) + \Gamma^{\alpha}_{bj}\Gamma^{b}_{ik} - \Gamma^{\alpha}_{bk}\Gamma^{b}_{ij}\right]$$

$$= -R^{\alpha}_{ijk}$$

i.e. $R^{\alpha}_{ijk} = -R^{\alpha}_{ikj}$

i.e. the curvature tensor is anti symmetric w.r. to the last two suffixes.

(iii) contraction $R^{\alpha}{}_{ijk}$

(a) Let us consider the first contraction setting $\alpha = i$ in (5)

$$R^{\alpha}_{\alpha jk} = \frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{\alpha k}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{\alpha j}\right) + \Gamma^{\alpha}_{bj}\Gamma^{b}_{\alpha k} - \Gamma^{\alpha}_{bk}\Gamma^{b}_{\alpha j}$$

$$(\alpha \leftrightarrow b)$$

$$= \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^k}\log\sqrt{g}\right) - \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^j}\log\sqrt{g}\right) \quad \left[\because \Gamma^{\alpha}_{\alpha k} = \frac{\partial}{\partial x^k}\log\sqrt{g}\right]$$

$$R^{\alpha}_{\alpha jk} = \frac{\partial^2}{\partial x^j \partial x^k}\left(\log\sqrt{g}\right) - \frac{\partial^2}{\partial x^k \partial x^j}\left(\log\sqrt{g}\right)$$

$$= 0$$

(b) Let us consider the confraction in R^{α}_{ijk} w.r. to α and j

$$R^{\alpha}_{i\alpha k} = \frac{\partial}{\partial x^{\alpha}}\left(\Gamma^{\alpha}_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{i\alpha}\right) + \Gamma^{\alpha}_{b\alpha}\Gamma^{b}_{ik} - \Gamma^{\alpha}_{bk}\Gamma^{b}_{i\alpha}$$

$$= \frac{\partial}{\partial x^{\alpha}}\left(\Gamma^{\alpha}_{ik}\right) - \frac{\partial^2}{\partial x^k \partial x^j}\left(\log\sqrt{g}\right) + \Gamma^{b}_{ik}\frac{\partial}{\partial x^b}\left(\log\sqrt{g}\right) - \Gamma^{\alpha}_{bk}\Gamma^{b}_{i\alpha} \qquad (10)$$

But $R^{\alpha}_{i\alpha k}$ or $R^{\alpha}_{i\alpha j} = R_{ij}$ is called the Ricci tensor which will very important in the field equation of the general theory of relativity.

2.2. Curvature tensor of the first kind R_{hijk}

The cristoffel curvature tensor of first kind is defined as

$$R_{hijk} = g_{h\alpha}R^{\alpha}_{ijk}$$

$$= g_{h\alpha}\left[\frac{\partial}{\partial x^j}\left(\Gamma^{\alpha}_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{ij}\right) + \Gamma^{\alpha}_{bj}\Gamma^{b}_{ik} - \Gamma^{\alpha}_{bk}\Gamma^{b}_{ij}\right]$$

$$= \frac{\partial}{\partial x^j}\left(g_{h\alpha}\Gamma^{\alpha}_{ik}\right) - \frac{\partial}{\partial x^k}\left(g_{h\alpha}\Gamma^{\alpha}_{ij}\right) - \Gamma^{\alpha}_{ik}\frac{\partial g_{h\alpha}}{\partial x^j} + \Gamma^{\alpha}_{ij}\frac{\partial g_{h\alpha}}{\partial x^k} + g_{h\alpha}\Gamma^{\alpha}_{bj}\Gamma^{b}_{ik} - g_{h\alpha}\Gamma^{\alpha}_{bk}\Gamma^{b}_{ij}$$

$$= \frac{\partial}{\partial x^j}\left(\Gamma_{ik,h}\right) - \frac{\partial}{\partial x^k}\left(\Gamma_{ij,h}\right) - \Gamma^{\alpha}_{ik}\left[\Gamma_{hj,\alpha} + \Gamma_{\alpha j,h}\right] + \Gamma^{\alpha}_{ij}\left[\Gamma_{hk,\alpha} + \Gamma_{\alpha k,h}\right] + g_{h\alpha}\Gamma^{\alpha}_{bj}\Gamma^{b}_{ik} - g_{h\alpha}\Gamma^{\alpha}_{bk}\Gamma^{b}_{ij}$$

$$\left[\because \Gamma_{ij,k} + \Gamma_{jk,i} = \frac{\partial g_{ik}}{\partial x^j} \text{ and } \Gamma^{i}_{jk} = g^{i\alpha}\Gamma_{jk,\alpha}\right]$$

$$= \frac{1}{2}\frac{\partial}{\partial x^j}\left[\frac{\partial g_{ih}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^h}\right] - \frac{1}{2}\frac{\partial}{\partial x^k}\left[\frac{\partial g_{ih}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h} - \Gamma^{\alpha}_{ik}g_{\alpha b}\Gamma^{b}_{hj} - \Gamma^{\alpha}_{ik}g_{bh}\Gamma^{b}_{\alpha j}\right.$$

$$\left. + \Gamma^{\alpha}_{ij}g_{\alpha b}\Gamma^{b}_{hk} + \Gamma^{\alpha}_{ij}g_{hb}\Gamma^{b}_{\alpha k} + \Gamma^{\alpha}_{bj}g_{h\alpha}\Gamma^{b}_{ik} - \Gamma^{\alpha}_{bk}g_{h\alpha}\Gamma^{b}_{ij}\right.$$

$$(\alpha \leftrightarrow b) \qquad (\alpha \leftrightarrow b)$$

$$= \frac{1}{2}\left[\frac{\partial^2 g_{ih}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{kh}}{\partial x^j \partial x^i} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{ih}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^i} + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^h}\right] - g_{\alpha b}\Gamma^{\alpha}_{ik}\Gamma^{b}_{hj} + g_{\alpha b}\Gamma^{\alpha}_{ij}\Gamma^{b}_{hk}$$

$\left[\because \Gamma^{\alpha}_{hk} = \Gamma^{\alpha}_{kh} and\ g_{h\alpha} = g_{\alpha h}\right]$

$$R_{hijk} = \frac{1}{2}\left[\frac{\partial^2 g_{ij}}{\partial x^k \partial x^h} + \frac{\partial^2 g_{kh}}{\partial x^j \partial x^i} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h} \frac{\partial g_{jh}}{\partial x^k \partial x^i}\right] - g_{\alpha b}\Gamma^{\alpha}_{ik}\Gamma^{b}_{hj} + g_{\alpha b}\Gamma^{\alpha}_{ij}\Gamma^{b}_{hk} \qquad (11)$$

Properties of the curvature fensor R_{hijk}

(i) $R_{iijk} = 0$

Proof : By definition

$$R_{iijk} = \frac{1}{2}\left[\frac{\partial^2 g_{ij}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ik}}{\partial x^i \partial x^j} \frac{\partial g_{ij}}{\partial x^i \partial x^k}\right] + g_{\alpha b}\Gamma^{\alpha}_{ik}\Gamma^{b}_{ij} - g_{\alpha b}\Gamma^{\alpha}_{ik}\Gamma^{b}_{ij} \quad [\because g_{\alpha b} = g_{b\alpha}]$$

$$= 0$$

$$\therefore R_{iijk} = 0$$

(ii) $R_{hijk} = -R_{ihjk}$

i.e. anti-symmetric w-r to the first two suffexes

Proof : By definition

$$R_{hijk} = \frac{1}{2}\left[\frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} + \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} \frac{\partial g_{jh}}{\partial x^k \partial x^i}\right] + g_{\alpha b}\Gamma^{\alpha}_{hk}\Gamma^{b}_{ij} - g_{\alpha b}\Gamma^{\alpha}_{jh}\Gamma^{b}_{ik} \qquad (12)$$

Interchanging i and h we get

$$R_{ihjk} = \frac{1}{2}\left[\frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} - \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{hk}}{\partial x^i \partial x^k}\right] + g_{\alpha b}\Gamma^{\alpha}_{hj}\Gamma^{b}_{ik} - g_{\alpha b}\Gamma^{\alpha}_{ij}\Gamma^{b}_{hk} \qquad (13)$$

$(\alpha \leftrightarrow b)$

Comparing (12) and (13) we get $R_{hijk} = -R_{ihjk}$

Similarly we can prove that

$$R_{hijk} = -R_{hikj}$$

(iii) $R_{hijk} = R_{jkhi}$ i.e. symmetric w.r.t two pair of suffexes

Proof : By definition

$$R_{hijk} = \frac{1}{2}\left[\frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} + \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} \frac{\partial^2 g_{jh}}{\partial x^k \partial x^i}\right] + g_{\alpha b}\Gamma^{\alpha}_{ij}\Gamma^{b}_{hk} - g_{\alpha b}\Gamma^{\alpha}_{ik}\Gamma^{b}_{hj} \qquad (14)$$

Also

$$R_{jkhi} = \frac{1}{2}\left[\frac{\partial^2 g_{kh}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ji}}{\partial x^k \partial x^h} - \frac{\partial^2 g_{ki}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^i}\right] + g_{ab}\Gamma^a_{kh}\Gamma^b_{ji} - g_{ab}\Gamma^a_{ki}\Gamma^b_{jh} \qquad (15)$$

$$(a \leftrightarrow b)$$

Since $g_{ab} = g_{ba}$, $\frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} = \frac{\partial^2 g_{ki}}{\partial x^j \partial x^h}$ and $\Gamma^a_{ij} = \Gamma^a_{ji}$

Comparing(14) and (15) we get

$$R_{hijk} = R_{jkhi}$$

(iv) $R_{hijk} + R_{hjki} + R_{hkij} = 0$

Proof : We have from (5)

$$R^a{}_{ijk} + R^a{}_{jki} + R^a{}_{kij} = 0$$

Considering the inner product of it with g_{ha} and summing over 'a'

$$g_{ha}R^a{}_{ijk} + g_{ha}R^a{}_{jki} + g_{ha}R^a{}_{kij} = 0$$

or, $\quad R_{hijk} + R_{hjki} + R_{hkij} = 0$

2.3 Bianchi Indentity :

By definition of curvature tensor of 2nd kind, we have

$$R^a_{ijk} \frac{\partial}{\partial x^j}\left(\Gamma^a_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^a_{ij}\right) + \Gamma^a_{bj}\Gamma^b_{ik} - \Gamma^a_{bk}\Gamma^b_{ij} \qquad (16)$$

Let us introduce a geodesic co ordinate system x^i in the n dimensional space V_n at the pole P_0

But at the pole P_0

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} \qquad \text{for } \Gamma_{ij,k} = 0 = \Gamma^i_{jk}$$

Considering x^i covariant derivative of (16) we get

$$R^a_{ijk,l} = \frac{\partial^2}{\partial x^l \partial x^j}\left(\Gamma^a_{ik}\right) - \frac{\partial^2}{\partial x^l \partial x^k}\left(\Gamma^a_{ij}\right) \qquad (17)$$

at the pole P_0

Also $R^{\alpha}_{ikl} = \frac{\partial}{\partial x^k}\left(\Gamma^{\alpha}_{il}\right) - \frac{\partial}{\partial x^l}\left(\Gamma^{\alpha}_{ik}\right) + \Gamma^{\alpha}_{bk}\Gamma^{b}_{il} - \Gamma^{\alpha}_{bl}\Gamma^{b}_{ik}$

$$\therefore R^{\alpha}_{ikl,j} = \frac{\partial^2}{\partial x^j \partial x^k}\left(\Gamma^{\alpha}_{il}\right) - \frac{\partial^2}{\partial x^j \partial x^l}\left(\Gamma^{\alpha}_{ik}\right) \quad (18)$$

at the pole P_0

and similarly we can get

$$R^{\alpha}_{ilj,k} = \frac{\partial^2}{\partial x^k \partial x^l}\left(\Gamma^{\alpha}_{il}\right) - \frac{\partial^2}{\partial x^k \partial x^j}\left(\Gamma^{\alpha}_{il}\right) \quad (19)$$

at the pole P_0

Now (17) +(18) + (19) gives

$$R^{\alpha}{}_{ijk,l} + R^{\alpha}{}_{ikl,j} + R^{\alpha}{}_{ilj,k} = 0 \quad (20)$$

This appears to be a tensor equation , but at every point we can construct the locally constant co ordinate system and hence it remains valid at every point of the system and so it is not an equation, but an indentity.This is called Bianchi identity

Now considering the inner product of (20) with $g_{h\alpha}$ and summing over 'α' we get

$$g_{h\alpha}R^{\alpha}{}_{ijk,l} + g_{h\alpha}R^{\alpha}{}_{ikl,j} + g_{h\alpha}R^{\alpha}{}_{ilj,k} = 0$$

or, $\left(g_{h\alpha}R^{\alpha}{}_{ijk}\right)_{,l} + \left(g_{h\alpha}R^{\alpha}{}_{ikl}\right)_{,j} + \left(g_{h\alpha}R^{\alpha}{}_{ilj}\right)_{,k} = 0 \qquad \left[\because g_{h\alpha,l} = 0\right]$

or $\quad R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0 \quad$ which is another from of Bianchi indentity

Ex. 1. Define Einstein tensor and show that it is divergence free.

Proof : From the Bianchi identity, we know

$$R^{\alpha}{}_{ijk,l} + R^{\alpha}{}_{ikl,j} + R^{\alpha}{}_{ilj,k} = 0$$

or, $\quad R^{\alpha}{}_{ijk,l} - R^{\alpha}{}_{ilk,j} + R^{\alpha}{}_{ilj,k} = 0 \qquad \left[\because R^{\alpha}{}_{ikl} = -R^{\alpha}{}_{ilk}\right]$

Let us consider a contraction w.r.t. 'α' and 'k'

$$R^{\alpha}{}_{ij\alpha l} - R^{\alpha}{}_{il\alpha,j} + R^{\alpha}{}_{ilj,\alpha} = 0$$

or, $\quad R_{ij,l} - R_{il,j} + R^{\alpha}{}_{ilj,\alpha} = 0 \qquad \left[\because R^{\alpha}{}_{ij\alpha} = R_{ij}\right]$

Considering inner product of it with g^{li} and summing over 'i' we get

$$g^{li}R_{ij,l} - g^{li}R_{il,j} + g^{li}R^{\alpha}{}_{ilj,\alpha} = 0$$

or, $\quad R^{l}{}_{j,l} - R_{,j} + R^{\alpha}{}_{j,\alpha} = 0$

$(l \to i) \quad (\alpha \to i)$

where $g^{ij}R_{ij} = R$ is the Einstein scalar

$$\text{or,}\quad R^{i}{}_{j,i} - R,_{j} + R^{i}{}_{j,i} = 0$$

$$\text{or,}\quad 2R^{i}{}_{j,i} - R,_{j} = 0$$

$$\text{or,}\quad R^{i}{}_{j,i} - \frac{1}{2}g^{i}{}_{j}R,_{i} = 0$$

$$\text{or,}\quad \left(R^{i}{}_{j} - \frac{1}{2}g^{i}{}_{j}R\right),_{i} = 0$$

$$\text{or,}\quad G^{i}{}_{j},_{i} = 0 \qquad (1)$$

$$\text{where}\quad G^{i}{}_{j} = R^{i}{}_{j} - \frac{1}{2}g^{i}{}_{j}R \qquad (2)$$

which is called the Einstein tensor, (1) shows that there arises the contraction with co variant derivative suffix and so it is called divergence free.

Ex. 2. If the differential equation $A_{i,j} = 0$ is integrable, they show that $R^{\alpha}{}_{ijk} = 0$

Solve : Given $A_{i,j} = 0$ is integrable

$$\text{or,}\quad \frac{\partial A_i}{\partial x^j} - A_\alpha \Gamma^{\alpha}_{ij} = 0$$

$$\text{or,}\quad \frac{\partial A_i}{\partial x^j} = A_\alpha \Gamma^{\alpha}_{ij} \qquad (1)$$

$$\text{or,}\quad \frac{\partial A_i}{\partial x^j} dx^j = A_\alpha \Gamma^{\alpha}_{ij} dx^j$$

$$\text{or,}\quad dA_i = A_\alpha \Gamma^{\alpha}_{ij} dx^j \quad \left[\because \frac{\partial v}{\partial x} dx = dv\right]$$

Integrating we get

$$A_i = \int A_\alpha \Gamma^{\alpha}_{ij} dx^j$$

This shows that the integrand in the R.H.S. must be a perfact differential of the from dB_i (say)

$$\therefore dB_i = A_\alpha \Gamma^{\alpha}_{ij} dx^j$$

$$\Rightarrow \frac{\partial B_i}{\partial x^j} dx^j = A_\alpha \Gamma^{\alpha}_{ij} dx^j$$

$$\Rightarrow \left(\frac{\partial B_i}{\partial x^j} - A_\alpha \Gamma^{\alpha}_{ij}\right) dx^j = 0$$

$$\Rightarrow \frac{\partial B_i}{\partial x^j} = A_\alpha \Gamma^\alpha_{ij} \left[\because dx^j \textit{is arbitrary}\right]$$

Differentiating partially w-r to x^k we get

$$\frac{\partial^2 B_i}{\partial x^k \partial x^j} = A_\alpha \frac{\partial}{\partial x^k}\left(\Gamma^\alpha_{ij}\right) + \frac{\partial A_\alpha}{\partial x^k}\Gamma^\alpha_{ij} \qquad (2)$$

Interchanging j and k in (2) we get

$$\frac{\partial^2 B_i}{\partial x^j \partial x^k} \; [illegible] \left(\Gamma^\alpha_{ik}\right) + \frac{\partial A_\alpha}{\partial x^j}\Gamma^\alpha_{ik} \qquad (3)$$

From (2) and (3 [illegible]

$$A_\alpha \frac{\partial}{\partial} [illegible] \frac{\partial A_\alpha}{\partial x^j}\Gamma^\alpha_{ik} - A_\alpha \frac{\partial}{\partial x^k} [illegible] - \frac{\partial A_\alpha}{\partial x^k}\Gamma^\alpha_{ij} = 0$$

$$\frac{\partial}{\partial x^j} [illegible] \Gamma^c_{\alpha j}\Gamma^\alpha_{ik} - A_\alpha \frac{\partial}{\partial x^k} [illegible] A_b \Gamma^b_{\alpha k}\Gamma^\alpha_{ij} = 0$$

$\alpha \to b$ [illegible] [uging (1)]

$$[illegible] \frac{\partial}{\partial x^j}\left(\Gamma [illegible] \frac{\partial}{\partial x^k}\left(\Gamma^b_{ij}\right) + \Gamma^b_{\alpha j}\Gamma [illegible] \Gamma^\alpha_{ij}\right] = 0$$

$$[illegible] \; A_b R^b_{ijk} = 0$$

$$[illegible] = 0 \qquad A_b \text{ is arbitrary}$$

[illegible] 3. A [illegible] essary and sufficient condition that V_n be locally flat (or flat) in the nbhd. of O [illegible] Riemann Christoffel tensor is zero.

Proof: Let V_n be locally flat in the nbhd. of O so that,

g_{ij} = constant, for all i and j

$$\Rightarrow \frac{\partial g_{ij}}{\partial x^k} = 0 \Rightarrow \Gamma^k_{ij} = 0, \text{ in the nbhd of O.}$$

Since R^α_{ijk} consists of products of Γ^k_{ij} and derivatives of Γ^k_{ij}. Hence $R^\alpha_{ijk} = 0$, in the nbhd. of O.

Conversely suppose that $R^\alpha_{ijk} = 0$, in the nbhd. of O. To prove that V_n is locally flat, it is enough to prove that g_{ij} = constant, in the nbhd. of O.

Given, co-ordinate system x^i, let us choose co-ordinate system x'^i s.t .

$$\Gamma^c_{ab} = \frac{\partial^2 x'^l}{\partial x^a \partial x^b}\frac{\partial x^c}{\partial x'^l} \qquad (1)$$

We know that

$$\Gamma'^{k}_{ij} = \Gamma^{c}_{ab} \frac{\partial x^{a}}{\partial x'^{i}} \frac{\partial x^{b}}{\partial x'^{j}} \frac{\partial x'^{k}}{\partial x^{c}} + \frac{\partial^2 x^{l}}{\partial x'^{i} \partial x'^{j}} \frac{\partial x'^{k}}{\partial x^{l}}.$$

Interchanging x^{i} and x'^{i}

$$\Gamma^{k}_{ij} = \Gamma'^{c}_{ab} \frac{\partial x'^{a}}{\partial x^{i}} \frac{\partial x'^{b}}{\partial x^{j}} \frac{\partial x^{k}}{\partial x'^{c}} + \frac{\partial^2 x'^{l}}{\partial x^{i} \partial x^{j}} \frac{\partial x^{k}}{\partial x'^{l}}$$

By (1) this becomes

$$\frac{\partial x^{k}}{\partial x'^{l}} \frac{\partial^2 x'^{l}}{\partial x^{i} \partial x^{j}} = \Gamma'^{c}_{ab} \frac{\partial x'^{a}}{\partial x^{i}} \frac{\partial x'^{b}}{\partial x^{j}} \frac{\partial x^{k}}{\partial x'^{c}} + \frac{\partial^2 x'^{l}}{\partial x^{i} \partial x^{j}} \frac{\partial x^{k}}{\partial x'^{l}}$$

$$\text{or,}\quad \Gamma'^{c}_{ab} \frac{\partial x'^{a}}{\partial x^{i}} \frac{\partial x'^{b}}{\partial x^{j}} \frac{\partial x^{k}}{\partial x'^{c}} = 0$$

Multiplying by $\frac{\partial x^{i}}{\partial x'^{l}} \frac{\partial x^{j}}{\partial x'^{m}} \frac{\partial x'^{p}}{\partial x^{k}}$ we get

$$\Gamma'^{c}_{ab} \delta^{a}_{l} \delta^{b}_{m} \delta^{p}_{c} = 0 \qquad \left[For \frac{\partial x^{i}}{\partial x^{j}} = \delta^{i}_{j} = \frac{\partial x'^{i}}{\partial x^{j}} \right]$$

$$\text{or}\quad \Gamma'^{p}_{lm} = 0$$

$$\text{or,}\quad \Gamma'^{k}_{ij} = 0 \qquad (2)$$

Thus $R'^{\alpha}_{ijk} = 0 \Rightarrow R^{\alpha}_{ijk} = 0$ (3)

Since if a tensor vanishes in one co-ordinate system then it vanishes in all systems. It follows that (1) is a solution of (3)

But (3) is given. Hence co-ordinate system x'^{i}, given by (1) exists

$$\therefore (2) \Rightarrow \frac{\partial g'_{ij}}{\partial x^{k}} = 0 \Rightarrow g'_{ij} = \text{constant}$$

Consequently a co-ordinate system x'^{i} exists relative to which g'_{ij} is constant. Hence V_n is locally flat.

Ex. 4 Show that the number of independent components of the curvature tensor of first kind

$$R_{hijk} \quad \text{is} \quad \frac{1}{12} n^2 (n^2 - 1)$$

Solution : Since the curvature tensor of the first kind R_{hijk} is a tensor of order 4 therefore in n dimensional space it has got n^4 compontns. But due to the following properties, this number is reduced considerably

(i) $R_{hijk} = -R_{ihjk}$

(ii) $R_{hijk} = -R_{hikj}$

(iii) $R_{hijk} = R_{jkhi}$

(iv) $R_{hijk} + R_{hjki} + R_{hkij} = 0$

Case (i) when there is one distinct suffix of the type R_{iiii}

$$R_{iiii} = -R_{iiii}$$

$$\Rightarrow 2R_{iiii} = 0 \quad \text{i.e. } R_{iiii} = 0$$

Case (ii) when there are two distinct suffixes of the type $R_{hihi}(h \neq i)$

The suffix 'h' can be assigned values out of n values in n different ways and no 'i' can be assigned values by the remaining $(n-1)$ ways

$\therefore$ They (both) can be assigned values in $n(n-1)$ ways.

$\therefore$ The no of independent components is $n(n-1)$. But due to the above properties this number will be reduced.

Now $R_{hihi} = -R_{ihhi} = R_{ihih}$ which is the more out come of inferchange of h and i.

Also $R_{hihi} = R_{hihi}$ the third symmetric which is identical.

Further

$$R_{hihi} + R_{hhii} + R_{hiih}$$

$$= R_{hihi} - R_{hihi} = 0$$

which is identically satisfy due to anti-symmetric property.

$\therefore$ The number $n(n-1)$ will be reduced to $\frac{1}{2}n(n-1)$

Case (iii) when there are three distinct suffixes of the type R_{hihj}

In this case the number of independent components (in general) $n(n-1)(n-2)$

But due to aforesaid properties this number will also be reduced.

Now, $\underbrace{R_{hihj} = -R_{ihhj}}_{antisymmetric} = -\underbrace{R_{hjih}}_{symmetric}$ (A)

But

$$R_{hihj} = -R_{hijh} \text{ (one antisymmetric property)}$$

$$= -R_{hjih} \left(i \leftrightarrow j\right) \quad \text{(B)}$$

The out come of one antisymmetric and the single symmetric property given by (A) is recoverable from the other antisymmetric property as given by (B). Hence number of independent component in this case

$$\frac{1}{2}n(n-1)(n-2)$$

Also

$$R_{hihj}+R_{hhji}+R_{hjih}=R_{hihj}+R_{ihhj}$$
$$=R_{hihj}-R_{hihj}=0$$

Hence cyclic property is identically satisfied and so there will be one reduction in the number of components due to this property.

Hence the number of idependent components with three distince suffixes is $\frac{1}{2}n(n-1)(n-2)$

Case (iv) when there are four distinct suffixes of the type R_{hijk}

But due to two anti symmetric and one symmetric properties the actual number $n(n-1)(n-2)(n-3)$ will reduced to $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}n(n-1)(n-2)(n-3)$. Besides there is the cyclic property viz.

$$R_{hijk}+R_{hjki}+R_{hkij}=0$$

or $R_{hijk}+R_{hjki}=-R_{hkij}$

This shows that, knowing two components the third can be known immediately.

$\therefore$ Due to this cyclic property the number $\frac{1}{8}n(n-1)(n-2)(n-3)$ will be further reduced to

$$\frac{2}{3}\times\frac{1}{8}n(n-1)(n-2)(n-3)$$
$$=\frac{1}{12}n(n-1)(n-2)(n-3)$$

The totel number of independent components

$$=0+\frac{1}{2}n(n-1)+\frac{1}{2}n(n-1)(n-2)+\frac{1}{12}n(n-1)(n-2)(n-3)$$
$$=\frac{1}{12}n(n-1)[6+6(n-2)+(n-2)(n-3)]$$

$$= \frac{1}{12} n(n-1)\left[6+6n-12+n^2-5n+6\right]$$

$$= \frac{1}{12} n(n-1)\left(n^2+n\right) = \frac{1}{12} n^2\left(n^2-1\right)$$

Exercise

1. Define Riemann Christoffel curvature tensor of 2nd kind and Discuss its properties

2. Show that the vanishing of Riemann curvature tensor is necessary and sufficient condition that the space be flat.

3. Establish Bianchi indentities

$$R^a{}_{ijk,l} + R^a{}_{ikl,j} + R^a{}_{ilj,k} = 0$$

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Unit 3
Theory of Gravitation

3.0 Introduction :

As soon as Einstein's general theory of relativity appeared as a viable theory of gravity, physicists started thinking for its modifications. The main reason, which aroused the tremendous interest, in this direction, was opening of a getway by general relativity to draw geometry into service to describe a physical field like gravitation.

History of science shows that most of the major advancements, in physics, have taken place through attempts for unification. Newtonian gravitational theory showed similarity between plantery motion and kinematics of projectiles on the earth. Quantization of radiation, by Einstein, explained aspects of photo-electric effect and removed theoretical difficulties in explaining the black body radiation and gave a right and successful direction to quantum theory. In the later part of nineteenth century, Maxwell unified electricity and magnetism, which explained the nature of light. Special theory of relativity came out of Maxwell's theory and basic principle of Newtonian mechanics.

3.1. The principle of Covariance :

In accordance with the principle of covariance the general laws of physics (nature) can be expressed in a form which is independent of the choice of space time co-ordinates. In other words the laws of physics remain covariant independent of the frame of reference.

The tensor form of the line element is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad (1)$$

where $g_{\mu\nu}$ is a fundamental covariant tensor of rank 2 obeying the law

$$\bar{g}_{ij} = \frac{\partial \bar{x}^i}{\partial x^\mu} \frac{\partial \bar{x}^j}{\partial x^\nu} g_{\mu\nu}$$

where the quantities carrying bar correspond to the new co-ordinate system.

An example that the tensor form of a law follows the general covariance principle, consider that a certain law of nature in a system of variable x is represented by the tensor equation

$$A^\mu_\nu = B^\mu_\nu \qquad (2)$$

Then the law when transformed to new system of variables $\bar{x}$ may be written as

$$\overline{A}_{\nu}^{\mu} - \overline{B}_{\nu}^{\mu} = \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \overline{x}^{\nu}} A_{j}^{i} - \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \overline{x}^{\nu}} B_{j}^{i}$$

$$\left(A_{j}^{i} - B_{j}^{i}\right) \frac{\partial \overline{x}^{\mu}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \overline{x}^{\nu}} = 0 \text{ using } (2)$$

$$\overline{A}_{\nu}^{\mu} = \overline{B}_{\nu}^{\mu} \qquad (3)$$

Equation (3) has exactly the same form as equation (2)

Thus the law of nature, when expressed in the form of tensor equation follows the general covariance principle. Hence according to general covariance principle laws of nature must be expressed in the form of tensorial equations.

3.2 Principle of Equivalence:-

The principle of equivalence gives specific expression to the correspondence between the results which would be obtained by an observer who makes measurements in a gravitational field using a frame of reference which is held stationary and the results obtained by a second observer who makes measurement in the absence of a gravitational field but using an accelerated frame of reference. In a qualitative way it is immediately evident that some measure of correspondence between the two sets of measurements should exists, since both observers would find an acceleration w.r.to their frames of reference for all free particles left to their own motion.

3.3 Some consequences of the principle of equivalence

3.3.1 Gravitational and inertial masses:-

The mass of a body is defined in two ways which differ from each other in a fundamental manner. There are two types of masses.

(i) The Kinematic or Inertial mass

(ii) The gravitational mass.

The inertial mass characterizes that inertial property of the body, while the gravitational mass characterizes the force with which the body is attracted according to law of universal gravitation.

Einstein appears to have found the clue for the principle of equivalence form the equality of the inertial mass (m_i) and gravitational mass (m_g) of an object.

If a force F acts on a body of inertial mass m_i and produces an acceleration a then we have from Newton's law of motion

$$F = m_i a \tag{1}$$

Now, if this body is dropped from the hand and if it falls with an acceleration a then according to Newton's law of gravitation.

$$m_i a = F = G\frac{M_g m_g}{R^2} \tag{2}$$

where M_g, m_g are the gravitational masses of the earth and the body (they are so called because they measure the gravitational effect involving the earth and the body) and R is the radius of the earth. Historically, Newton initially took the inertial mass and the gravitational mass to be different because there was apparently no physical reason why they should be taken to be the same.

However, in 1980, Newton performed an experiment with a pendulum (Figure 3.1) to decide the question. Considering the inertial mass m_i and the gravitational mass m_g of the

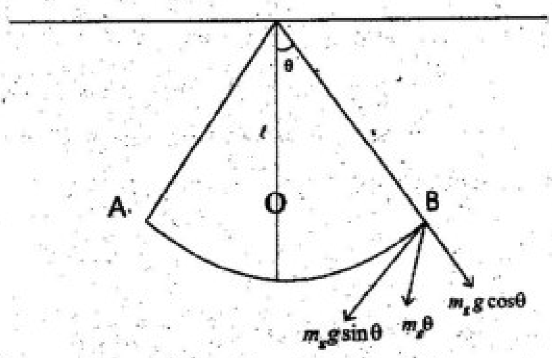


Fig.: 3.1

bob, the equation of motion, for a displacement, $\ell\sin\theta = \ell\theta$ (for all small θ) from the mean position O is

$$m_i\frac{d^2(l\theta)}{dt^2} = -m_g g\theta$$

$$\text{or, } \frac{d^2\theta}{dt^2} + \frac{m_g g}{m_i \ell}\theta = 0.$$

From equation (2) the period of oscillation is $T = 2\pi\sqrt{\frac{m_i l}{m_g g}}$ (3)

But Newton, on measurement of T, found that it agreed with the formula

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

This means that $m_i = m_g$ However, Newton failed to make out why it was so. Newton was followed by Bessel (1827), Eotvos (1890), southerns (1910), Dicke (1964) and Bragins (1971) who also arrived at the same result. Using this equality in equation (2) and writting $g = G\frac{M_g}{R^2}$ we obtain

$$\underset{\text{(inertial acceleration)}}{a} = \underset{\text{(gravitational field)}}{g} \qquad (4)$$

As already stated, equation (4) provided the clue for Einsteins principle of equivalence and inspired him to interpret gravitation as curvature of space time and demolish the body of absolute motion. One may therefore say that it is the principle of equivalence that is responsible for the equality of inertial mass and gravitational mass. Ironically, Newton missed it.

3.3.2 Effect of gravitational potential on the rate of a clock :

In accordance with the principle of equivalence there should be an agreement between the results obtained by a uniformity accelerated observer, who make measurement in the absence of any unitrinsic gravitational field, and those obtained in,similar experiments by a stationary observer in the presence of a uniform gravitational field. Since we can easily make approximate calculations as to the nature of the results obtained by the accelerated observer, this provides a simple method for investigating certain of the effects of gravity.

This method can be readily applied to determine the effect of differences in gravitational potential on the observed rate of clocks.

Let us first consider our observer in the absence of any intrinsic gravitational field who is subject to the constant acceleration 'g' and is provided with two indentically constructed clocks place one ahead of the other on a line parallel to the direction of acceleration at distance 'h' apart. Let the clock have the natural preriod T_0, and let light signal be sent at the end of each period from one clock to the other to permit a comparision of their observed rates.

Since the time necessary for a signal to pass between the two clocks will be approximately

$$t = \frac{h}{c}$$

where c is the velocity of light, the forward clock will the added velocity in the direction of motion.

$$v = gt = \frac{gh}{c}$$

in the interval necessary for light to pass from the near to the forward clock. hence by ordinary Doppler effect where that rate of the clocks are compared, the period of the near clock, where measured in terms of that of the forward clock with the helps of arriving light signals, will be found to be approximately

$$T = T_0\left(1+\frac{v}{c}\right) = T_0\left(1+\frac{gh}{c^2}\right) \qquad (1).$$

with the helps of the principle of equivalence however, this results can be re interpreted as also applying to the analogous situation of the two stationary clocks separated by a distance h in the direction of a uniform gravitational field of intensity 'g', so that we may immediately write as a consequence of (1)

$$T_2 = T_1\left(1+\frac{\Delta\psi}{c^2}\right) \qquad (2)$$

for the relation connecting the periods T_1 and T_2 of the two indentically constructed clocks with their difference in gravitational potential $\Delta\psi = gh$, the clock of the lower potential having the longer observed period.

3.3.3 The clock paradox :

The relation between the rate of a clock and its gravitational potential has also been found to furnish the solution for a well known paradox, which can arise when the behaviour of clocks is treated in accordance with the principle of special relativity without making due allowance for the principle of the general theory.

Consider two identically constructed clocks A and B, originally together and at rest, and let a force F_1 be then applied for a short time to clock B giving it the velocity u with which it then travels away from A at a constant rate for a time which is long compared with that necessary for a time which is long compared with that necessry for the acceleration. At the end of this time let a 2nd force F_2 be applied in the reverse direction, which bring B to rest and starts it back towards A with the reverse velocity (-u). Finally when it has returned to the neighbourhood of A, let the clock B be brought once more to rest by the action of a third force F_3.

Since by hypothesis the time intervals necessary for the acceleration and deceleration of clock B are made negligibly small compared with the time of travel at a constant velocity u.

By special theory of relativity, we have

$$\Delta T_A = \frac{\Delta T_B}{\sqrt{1-\frac{u^2}{c^2}}} = \Delta T_B\left(1+\frac{1}{2}\frac{u^2}{c^2}+....\right)$$

$$\therefore \Delta T_A > \Delta T_B \qquad (1)$$

with the idea of the relativity of all motion it should be equally acceptable to regard B as the clock which remains at rest and consider A as moving away with velocity -u and returning with the velocity +u then we have

$$\Delta T_B = \Delta T_A\left(1+\frac{1}{2}\frac{u^2}{c^2}+....\right)$$

$$\therefore \Delta T_B > \Delta T_A \qquad (2)$$

Now equation (2) contradicts equation (1). This is clock paradox.

Resolution of clock paradox :

The paradox arises because we have treated A and B at par with each other. But this is not correct,

$$P \xleftarrow{F_3\ a_3} \qquad \xrightarrow{-u\leftarrow} Q$$

$$(A,B)\rightarrow a_1 \qquad \leftarrow a_2$$

$$F_1 \qquad\qquad F_2$$

where as A remain at rest and is not acted upon by any forces, B is acted upon by three forces F_1, F_2 F_3 in order. So if this lack of parity between A and B is taken into account the paradox disappear.

Now while A remains at rest B experience acceleration a_1 due to F_1 at P, a raterdation a_2 due to F_2 at Q and finally a retardation a_3 due to F_3 at P.

So applying the principle of equivalence so far as B is concerned an accelerationg gravitational field g_1 occurs temporarily when B start from P a reverse gravitational field g_2 occurs temporarily when B apqroaches A and finally a retarding gravatational field g_3 occurs temporarily as B comes back at P.

Let us put

$$\Delta t_A = T_A + T_A' + T_A'' + T_A''' \qquad (3)$$

$$\Delta t_B = T_B + T_B' + T_B'' + T_B''' \qquad (4)$$

where T_A and T_B are the time measurement refered to the two clocks during which the clock A is now regarding as having the uniform velocity u and T_A', T_A'', T_A''' and T_B', T_B'', T_B''' are the time need for the three changes in the velocity of A and which are brought about at the begining, middle and the end of the experinlents by temporary introduction of appropriate gravitational field. g_1, g_2 and g_3 mentioned above.

Since the clock A is now moving we can write

$$T_B = \frac{T_A}{\sqrt{1-\frac{u^2}{c^2}}} = T_A\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

$$T_A = T_B\left(1-\frac{u^2}{c^2}\right)^{\frac{1}{2}}$$

$$= T_B\left(1-\frac{u^2}{2c^2}+...\right)$$

$$\therefore T_A = T_B\left(1-\frac{u^2}{2c^2}\right) \qquad (5)$$

Equation (5) is in contrast to the result obtain when B was taken as moving clock

Since the two clock will be at practically the same potential when the gravitational fields are introduced at the begining and at the end of the experiment, we can write

$$T_A' = T_B' \quad \text{and} \quad T_A''' = T_B''' \qquad (6).$$

On the other hand when the gravitational field is introduced at the middle of the experiment to produce the necessary reversal in the motion of A the two clocks will be a great distance from each other and we have

$$\therefore T_A'' = T_B''\left(1+\frac{g_2 d}{c^2}\right) \qquad (7)$$

Besides practically $\quad 2d = u \times T_B$

Since 2d is the total distance travel at the speed u.

Also $g_2 = \frac{2u}{T_B''}[\because v = ft]$ since $2u$ is the total change in velocity in time T_B''

using this result in (7)

$$\therefore T_A'' = T_B''\left(1+\frac{\frac{2u}{T_B''}\times u\times\frac{T_B}{2}}{c^2}\right)$$

$$= T_B'' + \frac{u^2}{c^2}T_B \qquad (8)$$

using this equation in (3) we get

$$\Delta T_A = T_A + T_A' + T_A'' + T_A'''$$

$$= T_B\left(1-\frac{u^2}{2c^2}+....\right) + T_B' + \left(T_B'' + \frac{u^2}{c^2} + T_B\right) + T_B'''$$

$$= T_B\left(1+\frac{u^2}{2c^2}+....\right) + T_B' + T_B'' + T_B'''$$

since T_B', T_B'', T_B''' are very short (small) compared with T_B so we can neglect the primed quantities.

$$\Delta T_A = T_B\left(1+\frac{u^2}{2c^2}+....\right)$$

$$\Delta T_A = T_B\left(1+\frac{u^2}{2c^2}\right)\left[\because \frac{u^2}{c^2}\text{ is very small}\right]$$

Since $T_B >> T_B', T_B'', T_B'''$

Also $\Delta T_B = T_B$

$$\Delta T_A = \Delta T_B\left(1+\frac{u^2}{2c^2}\right) \qquad (9)$$

Comparing this result with equation (1) we see that whether we consider A or B to be the clock, which moves we obtain the same expression for the relative readings of the two clocks.

The solution thus provided for the well known clock paradox of the special theory gives a specially illuminating examples of the justification for regarding all kinds of motion as relative that has been made possible by the adoption of the general theory of relativity.

3.4 Material Energy Tensor (or Energy Momentum Tensor) :

If ρ_0 is the proper density of matter and $\frac{dx^\mu}{ds}$ refers to the motion of the matter, then in relativistic units the material energy tensor is defined as

$$T^{\mu\nu} = \rho_0 \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \qquad (1)$$

In general any of the tensor contravariant tensor $T^{\mu\nu}$ its associated mixed tensor T^μ_ν and co variant tensor $T_{\mu\nu}$ is called as meterial energy tensor or energy momentum tensor or energy tensor of matter.

In Galilean co ordinate system, we have

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \text{ where } c = 1$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = 1 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]$$

Taking $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = v^2$

We have

$$\left(\frac{ds}{dt}\right)^2 = 1 - v^2 \qquad (2)$$

If ρ is the co ordinate density of matter moving with velocity v relative to Galilean co ordinates then

$$\rho = \frac{\rho_0}{1 - \frac{v^2}{c^2}} = \frac{\rho_0}{1 - v^2} \text{ in relativistic units}$$

$$\Rightarrow \rho_0 = \rho(1 - v^2) \qquad (3)$$

using (2) and (3) we have

$$\rho_0 = \rho\left(\frac{ds}{dt}\right)^2 \qquad (4)$$

Hence in Galilean co-ordinate (1) becomes

$$T^{\mu\nu} = \rho\left(\frac{ds}{dt}\right)^2 \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

$$= \rho \frac{dx^\mu}{ds}\frac{ds}{dt}\frac{dx^\nu}{ds}\frac{ds}{dt} = \rho \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \qquad (5)$$

If we write

$\frac{dx^1}{dt}=u, \frac{dx^2}{dt}=v, \frac{dx^3}{dt}=w$ then (5) becomes

$$T^{\mu\nu}=\begin{bmatrix} \rho u^2 & \rho uv & \rho uw & \rho u \\ \rho uv & \rho v^2 & \rho vw & \rho v \\ \rho uw & \rho vw & \rho w^2 & \rho w \\ \rho u & \rho v & \rho w & \rho \end{bmatrix} \quad (6)$$

In atomically constituted matter, a volume which is regarded as small for macroscopic treatment contains particles with widely varying motions. Thus the equation (6) should be summed up for varying motion of the particles. For this we have to added to (6) the tensor formed by the internal stresses i.e.

$P_{\alpha\beta}(\alpha,\beta=x,y,z)$

Hence for atomically constituted matter, we have

$$T^{\mu\nu}=\begin{bmatrix} p_{xx}+\rho u^2 & p_{xy}+\rho uv & p_{xz}+\rho uw & \rho u \\ p_{xy}+\rho uv & p_{yy}+\rho v^2 & p_{yz}+\rho vw & \rho v \\ p_{xz}+\rho uw & p_{yz}+\rho vw & p_{zz}+\rho w^2 & \rho w \\ \rho u & \rho v & \rho w & \rho \end{bmatrix} \quad (7)$$

where ρ represents the whole density and u, v, w. the avarage or mass motion of macroscopic elements.

Now consider the equation

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}}=0 \quad (8)$$

Taking first $\mu=4$ and using (7) we get

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}+\frac{\partial\rho}{\partial t}=0 \quad (9)$$

which represents the equation of continuity in hydrodynamics.

Now taking $\mu=1$ and using (7)

$$\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{xy}}{\partial y}+\frac{\partial p_{xz}}{\partial z}=-\left[\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}+\frac{\partial(\rho u)}{\partial t}\right]$$

$$=-u\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}+\frac{\partial(\rho u)}{\partial t}\right]-\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}+\frac{\partial u}{\partial t}\right]$$

using (9) we have

$$= -\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} + \frac{\partial u}{\partial t}\right]$$

$$= -\rho\frac{du}{dt} \qquad (10a)$$

where $\frac{d}{dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$.

similarly for $\mu = 2,3$ equation (8) gives

$$\frac{\partial p_{yz}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{yz}}{\partial z} = -\rho\frac{dv}{dt} \qquad (10b)$$

$$\frac{\partial p_{yz}}{\partial x} + \frac{\partial p_{zy}}{\partial y} + \frac{\partial p_{zz}}{\partial z} = -\rho\frac{dw}{dt} \qquad (10c)$$

Here $\frac{du}{dt}$, $\frac{dv}{dt}$ and $\frac{dw}{dt}$ represents the components of the acceleration of the element of the fluid.

Equation (10a) and (10c) are well known equations of motion in hydrodynamics in the absence of any external forces.

Equation (9) and (10) express directly the conservation of mass and momentum so that in Galilean co ordinates the principles of conservation of mass and momentum are contained in equation

viz. $\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0$ At $\Gamma^{\alpha\nu}_{\nu} = 0$ relative to Galilean co ordinates

Therefore $T^{\mu\nu}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\nu}}$

Hence in Galilean co ordinate system the equation (8) may be expessed as

$$T^{\mu\nu}_{;\nu} = 0 \qquad (11)$$

This equation represents the divergence of $T^{\mu\nu}$ is zero. In fact $\frac{\partial T^{\mu\nu}}{\partial x^{\nu}}$ represents the rate of creation of mass and momentum in unit valume.

3.5 Energy momentum tensor in case of perfact fluid :

In the case of perfact fluid which we define as a mechanical medium incapable of exerting transverse stresses, the only components of stress for a local observer will be those corresponding to the proper hydrostatic pressure p_0 so that the energy momentum tensor will then have in proper co ordinates the simple set of components

$$T_0^{11} = T_0^{22} = T_0^{33} = p_0, T_0^{44} = \rho_0 \qquad (1)$$

$$T_{\alpha\beta} = 0, \quad \alpha \neq \beta$$

where p_0 and ρ_0 denotes repectively the pressure and density of a proper fluid in proper co-ordinate system.

In proper co ordinate system Galilean co-ordinate system holds for which

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2$$

where $c = 1$ i.e. the motion of fluid is considered in gravitational system.

Let g_o^{ij} denote fundamental tensor in Galilean co ordinate system,

so that

$$g_o^{11} = g_0^{22} = g_0^{33} = -1 \qquad g_o^{44} = 1, g^{ij} = 0 \ \ for\ i \neq j$$

Let T^{ij} and g^{ij} respectively denoted the energy momentum tensor and fundamental tensor in arbitrary co ordinate system. By tensor law of transformation.

$$T^{ij} = T_0^{ab} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^b}$$

$$= \sum_{a=1}^{4} T_0^{aa} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^a} \qquad \text{[using (1)]}$$

$$= p_0 \sum_{a=1}^{3} T_0^{aa} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^a} + \rho_0 \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^a} \qquad (2) \quad \text{[again using (1)]}$$

and

$$g^{ij} = g_0^{ab} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^b}$$

$$= \sum_{a=1}^{4} g_0^{aa} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^a}$$

$$= -\sum_{a=1}^{3} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^a} + \frac{\partial x^i}{\partial x_0^4} \frac{\partial x^j}{\partial x_0^4}$$

$$\Rightarrow \sum_{a=1}^{3} \frac{\partial x^i}{\partial x_0^a} \frac{\partial x^j}{\partial x_0^a} = -g^{ij} + \frac{\partial x^i}{\partial x_0^4} \frac{\partial x^j}{\partial x_0^4}$$

using this result in (2)

$$T^{ij} = p_0 \left[-g^{ij} \frac{\partial x^i}{\partial x_0^4} \frac{\partial x^j}{\partial x_0^4} \right] + \rho_0 \frac{\partial x^i}{\partial x_0^4} \frac{\partial x^j}{\partial x_0^4}$$

$$= (p_0 + \rho_0)\frac{\partial x^i}{\partial x_0^4}\frac{\partial x^j}{\partial x_0^4} - p_0 g^{ij} \qquad (3)$$

Since the fluid is at rest in the proper co ordinate system and hence the velocity components can be taken as

$$\frac{dx_0^1}{ds} = \frac{dx_0^2}{ds} = \frac{dx_0^3}{ds} = 0, \frac{dx_0^4}{ds} = 1 \qquad (4)$$

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial x_0^j}\frac{dx_0^j}{ds} = \frac{\partial x^i}{\partial x_0^4}\frac{dx_0^4}{ds} = \frac{\partial x^i}{\partial x_0^4} \quad \text{[using (4)]}$$

using this result in (3) we have

$$T^{ij} = (p_0 + \rho_0)\frac{dx^i}{ds}\frac{dx^j}{ds} - p_0 g^{ij}$$

Taking $p_0 = p$ and $\rho_0 = \rho$ we get

$$T^{ij} = (p + \rho)\frac{dx^i}{ds}\frac{dx^j}{ds} - pg^{ij}$$

$$= (\rho + p)v^i v^j - pg^{ij}$$

where $v^i = \frac{dx^i}{ds} =$ velocity component

This can also be expressed as

$$T_\nu^\mu = (\rho + p)v^\mu v_\nu - pg_\nu^\mu \qquad \text{or,} \qquad T_{\mu\nu} = (\rho + p)v_\mu v_\nu - pg_{\mu\nu}$$

Exercise

1. In general relativity derive the expression for the energy momentum tensor T^{ij} for a perfect fluid distribution in the from

$$T^{ij} = (\rho + p)v^i v^j - pg^{ij}$$

2. Define material energy tensor. Show that in Galilean co ordinates.

$$T^{ij} = \rho\frac{dx^i}{ds}\frac{dx^j}{ds}$$

3. Explain the principle of (i) Equivalence (ii) Covariance

4. What is clock paradox? Discuss the resolution of clock paradox in general theory of relativity.

●●●

Unit-4

The Gravitational Fluid in Empty Space in Presence of Matter and Energy

4.0 Introduction

There is a major difference in characteristic features of electro-magnetic and gravitational fields. Electromagnetic fields do not carry any charge and do not interact with itself. So electromagnetic field equations are linear. Gravitational fields, given by g_{ij}, have self interaction. Moreover, there is energy momentum exchange between matter and gravitation. These properties of gravitational fields lead to non-linearity of field equations.

4.1 Einstein Field Equations:

Einstein did not derive the field equation but wrote down on the basis of certain considerations as listed below-

(i) In first place, according to the principle of covariance, the field equation must be a tensor equation.

(ii) In the second place, there is the well-known poisson law in gravitation.

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=4\pi G\rho \qquad (1)$$

where ψ is the gravitational potential, ρ is the density of matter distribution and G Newton's gravitation constant. Under appropriate conditions the new law of gravitation must yield an equation of the form, equation (i) If the metric functions $g_{\mu\nu}\left(g_{\mu\nu}\text{ being symmetric}\right)$ of Einstein theory are to correspond to a single function ψ of Newton's theory, then according to equation (1), the new law of gravitation must not contain higher dirivatives of $g_{\mu\nu}$ then the second. The Rioci tensor $R_{\mu\nu}$ indeed does not contain higher derivatives of $g_{\mu\nu}$ then the second. The left-hand side of equation (i) Should therefore be related to $R_{\mu\nu}$ in an appropriate form.

(iii) In the third place, the right-hand side of equation (i) containing the density of matter distribution should be related to $T_{\mu\nu}$ the energy momentum tensor, of Einstein theory, according to special relativeity, the conservation law in flat space-time is

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0 \quad (2)$$

Obviously equation (2) would be valid in a geodesic co-ordinate system. In this co-ordinate system

$$T^{\mu}_{\alpha\beta} = 0 \quad (3)$$

Hence using equations (2) and (3) we arrive at the result

$$T^{\mu\nu}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\alpha\nu} T^{\alpha\nu} + \Gamma^{\mu}_{\alpha\nu} T^{\mu\alpha} = 0 \quad (4)$$

Equation (4) is a tensor equation holding in the geodesic co-ordinate system. Hence it should be true in any other co-ordinate system. So equation (4) holds in a curved space time.

(iii) Finally, i.e. in the fourth place, $R_{\mu\nu}$ occurs in an expression.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \text{ such that}$$

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)_{;\nu} = 0 \text{ [Bianchi identity]}$$

The above four considerations (i– iv) inspired Einstein to write down, as the general relativistic analogue of the poisson equation, the law of gravitation in the following form:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \ R = -kT_{\mu\nu} \quad (6)$$

where k is a constant. The validity of equation (6) would be clear when we work out the Newtonian approximation from this theory in the next section. We also obtain the value of k and find justification for the minus sign.

Since $\left(g^{\mu\nu}\right)_{;\nu} = 0$ we can write equation (5) in the following enlarged form with a constant Λ

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} \right)_{;\nu} = 0 \quad (7)$$

On cosmological ground. Then we can write the law of gravitation in the following form also:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -kT_{\mu\nu} \quad (8)$$

If we consider gravitation in empty space due to some source then from equation (6) we get

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \qquad (9)$$

On contraction we have

$$R - \frac{1}{2} \times 4 \times R = 0 \qquad (10)$$

Using equation (10) in equation (9) we have the law of gravitation in empty space due to a source in the following form.

$$R_{\mu\nu} = 0 \qquad (11)$$

4.2 Newton's equation of motion as an approximation of geodesic equation-

We shall now like to investigate as to what is the trajectory of a free particle in the space time manifold, the geometry of which is specified by the line element.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad (1)$$

We shall obtain the trajectory by demanding that the trajectory between two points A and B of the space time continuum be such that

$$\partial \int_A^B ds = 0 \qquad (2)$$

This leads to the geodesic equation of motion of the test particles

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \qquad (3)$$

where $\Gamma^\mu_{\alpha\beta}$'s are the christoffel's symbols of the 2nd kind and are given by

$$\Gamma^\mu_{\alpha\beta} = g^{\mu\nu} \Gamma_{\alpha\beta,\nu}$$

with $$\Gamma_{\alpha\beta,\nu} = \frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right).$$

Now in the special relativity the line element corresponds to a Euclidian space time (flat space time manifold) and all the $g_{\mu\nu}$'s are constant and independent of the co-ordinates. Consequently all the $\Gamma^\mu_{\alpha\beta}$'s vanish and in that case covariant equation of the trajectory given by (3) reduces to the equation of a straight line i.e. $$\frac{d^2x^\mu}{ds^2}=0 \tag{4}$$

It is remarkable that the metric tensor $g_{\mu\nu}$ determines both the geometry of the space time continuum and also the trajectory. In the first case, the components of the metric tensor $g_{\mu\nu}$ determine the structure of the geometry while in the case of trajectory it is the first derivative of $g_{\mu\nu}$'s with respect to the co-ordinates, which make their appearance via the christoffel's three index symbols. Looked at the equation of trajectory in the context, the $g_{\mu\nu}$'s play the role of gravitational potential in analogy to the Newton's equation of motion.

Now the equation arises, what is the connection of the geodesic equation given by (1) with Newton's equation of motion? For this we recall that for Euclidian space the components of the metric tensor are all constants and are given by

$$g_{\mu\nu}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \tag{5}$$

Since $ds^2=g_{\mu\nu}dx^\mu dx^\nu$ in Riemannien space

$=-dx^2-dy^2-dz^2+c^2dt^2$ in Euclidian space

Since the christoffel's symbols of both kinds are zero, equation (3) reduces to equation (4). Let us now assume that the $g_{\mu\nu's}$ are not constant, but differ from the values given by (5) by infinitesimal amount viz in a weak static field

$$g_{\mu\nu}=\eta_{\mu\nu}+\psi_{\mu\nu} \tag{6}$$

where $\eta_{\mu\nu}$ is the flat space time metric co-efficient and $|\psi_{\mu\nu}|<<1$ (i.e. $\psi_{\mu\nu}$ are small quantifies) and we can neglect terms of second and higher orders in $|\psi_{\mu\nu}|$

We obtain by supposing field as static $\left(i.e.\ \frac{\partial g_{\mu\nu}}{\partial x^4}=0\right)$

$$\therefore \Gamma^\alpha_{44}=g^{\alpha\alpha}\Gamma_{44,\alpha}$$

$$=\frac{1}{2g_{\alpha\alpha}}\left(-\frac{\partial g_{44}}{\partial x^\alpha}\right);\qquad \alpha=1,\ 2,\ 3$$

$$=\frac{1}{2g_{\alpha\alpha}}\left[-\frac{\partial}{\partial x^\alpha}(1+\psi_{44})\right]$$

$$= \frac{1}{2(\eta_{\alpha\alpha} + \psi_{\alpha\alpha})}\left[-\frac{\partial}{\partial x^\alpha}(1+\psi_{\alpha\alpha})\right]$$

$$= \frac{1}{2(-1+\psi_{\alpha\alpha})}\left[-\frac{\partial}{\partial x^\alpha}(1+\psi_{44})\right]$$

$[\because \eta_{11} = \eta_{22} = \eta_{33} = -\eta_{44}$ and $\eta_{\mu\nu} = 0 = g_{\mu\nu}$ for $\mu \neq \nu]$

$$\cong \frac{1}{2}\frac{\partial \psi_{44}}{\partial x^\alpha} \tag{7}$$

In Galilean co-ordinate system $x^1 = x, x^2 = y, x^3 = z, x^4 = ct$

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2dt^2$$

$$= c^2\left(1 - \frac{v^2}{c^2}\right) dt^2 \tag{8}$$

For small velocities $\frac{v}{c} << 1$

$$ds^2 \cong c^2dt^2 = (dx^4)^2$$

$$\therefore ds \cong dx^4 = cdt \tag{9}$$

Hence in weak static field i.e. if does not change with time the velocity components can be taken as $\frac{dx^1}{ds} = 0 = \frac{dx^2}{ds} = \frac{dx^3}{ds}, \frac{dx^4}{ds} = 1$

By virtue of above equation (3) becomes

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{44}\frac{dx^4}{ds}\frac{dx^4}{ds} = 0, \quad i = 1,\ 2,\ 3$$

$$\Rightarrow \frac{d^2x^i}{ds^2} = -\Gamma^i_{44}$$

using (7) we get

$$\frac{d^2x^i}{ds^2} = -\frac{1}{2}\frac{\partial \psi_{44}}{\partial x^i}$$

using (9) this equation may be written as

$$\frac{d^2x^i}{dt^2} = -c^2\Gamma^i_{44} = -\frac{c^2}{2}\frac{\partial \psi_{44}}{\partial x^i} \tag{10}$$

$i = 1, 2, 3$

Newton's equation of motion are

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i} \tag{11}$$

where ϕ is the potential function

From (10) and (11) we have

$$-\frac{\partial}{\partial x^i}\left(\frac{1}{2}c^2\psi_{44}\right) = -\frac{\partial \phi}{\partial x^i}$$

Integrating we get

$$\int \frac{\partial \psi_{44}}{\partial x^i}\ dx^i = \frac{2}{c^2}\int \frac{\partial \phi}{\partial x^i}\ dx^i + \text{ constant}$$

$$\Rightarrow \psi_{44} = \frac{2\phi}{c^2} + k_1$$

$$\Rightarrow 1 + \psi_{44} = \frac{2\phi}{c^2} + \text{constant}$$

$$\Rightarrow g_{44} = \frac{2\phi}{c^2} + k$$

Since flat space, $g_{44} = 1,\ \phi = 0$

$$\therefore k = 1$$

Where the constant has been so chosen as to make the potential ϕ vanish at a great distance from the gravitating bodies, where

$$g_{44} \rightarrow 1$$

$$\therefore g_{44} = 1 + \frac{2\phi}{c^2}$$

Hence the geodesic equation are reducible to Newton's equation of motion in case of weak static field if $g_{44} = 1 + \frac{2\phi}{c^2}$

Thus Newton's theory of gravitation can be regarded as the first approximation to the general theory with the quantity g_{44} of the general theory closely related to the gravitational potential ϕ of the Newtonian theory.

4.3 Poisson equation as an approximation of Einstein's field equation:

Assuming cosmological constant Λ to be very small quantity, Einstein field equations are-

$$R_{\mu\nu} - \frac{1}{2}\ g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{1}$$

In order to obtain Newtonian approximation of above equation, consider the motion of the test particle in a very weak static field which in characterized by

$$g_{\mu\nu} = \in_{\mu\nu} + \eta_{\mu\nu} \qquad (2)$$

where $\in_{\mu\nu}$ is the metric tensor for Euclidian space and $\eta_{\mu\nu}$ is the function of (x, y, z) and is so small that the powers of $\eta_{\mu\nu}$ higher than the first are neglected. In this case, (for weak static field) the line element will differ very slightly from that of the special relativity and we must have

$$\left.\begin{array}{l} \in_{11} = \in_{22} = \in_{33} = -\in_{44} = -1 \\ \in_{\mu\nu} = g_{\mu\nu} = 0, \ \mu \neq \nu \end{array}\right\} \qquad (3)$$

Since the field is static i.e. it does not change with time and hence velocity components can be taken

$$\frac{dx^1}{ds} = \frac{dx^2}{ds} = \frac{dx^3}{ds} = 0, \frac{dx^4}{ds} = 1 \qquad (4)$$

The co-ordinates considered are Galilean co-ordinates

$$\therefore x^1 = x, x^2 = y, x^3 = z, x^4 = ct$$

The geodesic equation's are reduced to Newtonian equations of motion if

$$g_{44} = 1 + \frac{2\psi}{c^2} = 1 + 2\psi, \text{ taking } c = 1$$

The components of energy momentum tensor are

$$T^{\mu\nu} = \rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}$$

All the components of energy tensor (in the limit of Newtonian approximation) will be approximately zero separately, except.

$$T^{44} = \rho \frac{dx^4}{ds} \frac{dx^4}{ds} = \rho$$

$$T = g^{\mu\nu} T_{\mu\nu} = g^{44} T_{44}$$

$$= \frac{1}{g_{44}} T_{44}$$

$$= (1 + \eta_{44})^{-1} T_{44} \cong \rho$$

From the field equation (1) we have

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = -8\pi g^{\mu\nu}T_{\mu\nu}$$

$$\Rightarrow R - 2R = -8\pi \; T$$

$$\Rightarrow R = 8\pi T$$

$$\therefore R_{\mu\nu} = -8\pi T_{\mu\nu} + \frac{1}{2}g_{\mu\nu} 8\pi T$$

$$= -8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$$

$$R_{44} = -8\pi\left[T_{44} - \frac{1}{2}\; g_{44}T\right]$$

$$= -8\pi\left[\rho - \frac{1}{2}g_{44}\rho\right]$$

$$= -8\pi\rho\left[1 - \frac{1}{2}g_{44}\right] = -4\pi\rho \tag{5}$$

Since

$$R_{\mu\nu} = \frac{\partial}{\partial x^{\lambda}}\;\Gamma^{\lambda}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda}$$

From which we get

$$R_{44} = \frac{\partial}{\partial x^{4}}\;\Gamma^{\lambda}_{4\lambda} - \frac{\partial}{\partial x^{\lambda}}\Gamma^{\lambda}_{44} + \Gamma^{\alpha}_{4\lambda}\Gamma^{\lambda}_{\alpha 4} - \Gamma^{\alpha}_{44}\Gamma^{\lambda}_{\alpha 4}$$

Writing upto first order approximation, we have,

$$R_{44} = \frac{\partial}{\partial x^{4}}\;\Gamma^{4}_{4\lambda} - \frac{\partial}{\partial x^{\lambda}}\Gamma^{\lambda}_{44}$$

Since $g_{\mu\gamma}$ are not the function of time in static gravitational field

i.e. $\frac{\partial g_{44}}{\partial x^4} = 0$

$$\therefore R_{44} = -\frac{\partial}{\partial x^{\lambda}}\;\Gamma^{\lambda}_{44} = 0 \tag{6}$$

Also, $\frac{\partial}{\partial x^4}\;\Gamma^{4}_{44} = 0$, Since $\frac{\partial g_{44}}{\partial x^4} = 0$ for all values of μ and ν and $g_{4\alpha} = 0$

For $\alpha = 1, 2, 3$

$$\therefore R_{44} = -\frac{\partial}{\partial x^{\alpha}}\;\Gamma^{\alpha}_{44} \quad \alpha = 1,\; 2,\; 3 \tag{7}$$

From equations (5) and (7) we get

$$\frac{\partial}{\partial x^{\alpha}} \Gamma^{\alpha}_{44} = 4\pi\rho \qquad (8)$$

Also for weak static field $\alpha = 1, 2, 3$

We have,

$$\Gamma^{\alpha}_{44} = g^{\alpha b}\Gamma_{44,b} = g^{\alpha\alpha}\Gamma_{44,\alpha} = \frac{1}{g_{\alpha\alpha}}\Gamma_{44,\alpha}$$

$$= \frac{1}{(-1+\eta_{\alpha\alpha})}\left(-\frac{1}{2}\frac{\partial g_{44}}{\partial x^{\alpha}}\right)$$

$$= (1-\eta_{\alpha\alpha})^{-1}\left(\frac{1}{2}\frac{\partial g_{44}}{\partial x^{\alpha}}\right)$$

Now (8) is expressible as

$$\frac{\partial}{\partial x^{\alpha}}\left[\frac{1}{2}\frac{\partial g_{44}}{\partial x^{\alpha}}\right] = 4\pi\rho$$

$$\Rightarrow \sum_{\alpha=1}^{3}\frac{\partial^2 g_{44}}{(\partial x^{\alpha})^2} = 8\pi\rho$$

$$\Rightarrow \frac{\partial^2 g_{44}}{\partial x^2} + \frac{\partial^2 g_{44}}{\partial y^2} + \frac{\partial^2 g_{44}}{\partial z^2} = 8\pi\rho$$

$$\Rightarrow \nabla^2 g_{44} = 8\pi\rho$$

$$\Rightarrow \nabla^2(1+2\psi) = 8\pi\rho$$

$$\Rightarrow \nabla^2\psi = 4\pi\rho$$

Which is poisson's equation in relativistic units. Thus the relativistic general theory of gravitations corresponds to the Newton's theory of gravitation in the presence of matter in the non relativistic limit of a weak, static gravitational field.

4.4. Solution of Einstein gravitational equations in empty space. Schwarzschild exterior solution for the gravitational field of an isolated particle :

The first exact solution of the Einstein equation was obtained by K. Schwarzschild (1916) for static and spherically symmetric field which is a good approximation for the gravitational field of the sun.

In empty space the law of gravitation chosen by Einstein

$$R_{\mu\nu} = 0 \text{ (in empty space)} \qquad (1)$$

Einstein modified equation (1) later on and cosmological constant Λ was included and took

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \text{ (as field equation in empty space)} \qquad (2)$$

The solution of the above equations consists of finding the line element for interval in the empty space surrounding a gravitating point particle, which ultimately corresponds to the field of an isolated particle continually rest at the origin. This solution was first given by Schwarzschild and is of great importance. Since it provides a treatment of the gravitational field surrounding the sun far use discussing three crucial tests that distinguish between the predicting of Newtonian theory of gravitation and the more exact predictions of the theory of relativity.

In absence of mass (empty space) the space time would be flat so that the line element in spherical polar co-ordinate would be expressed as

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2 \text{ (flat space)} \qquad (3)$$

The presence of gravitational mass modify the space time. However since the mass is static and isolated the line element would spatially spherically symmetric about the point mass. The most general form of a such line element may be expressed as

$$ds^2 = -e^{2\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + e^{2\nu} dt^2 \text{ (curve space)} \quad (4)$$

With $x^1 = r,\ x^2 = \theta,\ x^3 = \phi,\ x^4 = t$ and λ and ν are functions of r only. since for spherically symmetric isolated particle the field will depend on r alone not on θ and ϕ

Since the gravitational field due to a particle diminishes as we go on infinite distance. Hence the lone element (4) reduces to Galilean line element (3) at an infinite distance from the particle.

$$\therefore \lambda \to 0,\ \nu \to 0, \text{ as } r \to \infty$$

$$\therefore g_{\mu\nu} = \begin{bmatrix} -e^{2\lambda} & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & e^{2\nu} \end{bmatrix} \qquad (5)$$

$$g_{\mu\nu} = |g_{\mu\nu}| = -r^4 \sin^2\theta \ e^{2(\lambda+\nu)} \qquad (6)$$

The elements of matrix corresponding to the inverse matrix tensor are

$$g^{\mu\nu} = \frac{Co - factor\ \ of\ \ g_{\mu\nu}}{g} \qquad (7)$$

$$g_{\mu\nu} = \begin{bmatrix} -e^{-2\lambda} & 0 & 0 & 0 \\ 0 & -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2\sin^2\theta} & 0 \\ 0 & 0 & 0 & e^{-2\nu} \end{bmatrix} \qquad (8)$$

The surviving christoffel's symbols calculated by using the formula viz.

$$\Gamma_{\mu\nu,\alpha} = \frac{1}{2}\left[\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} + \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right]$$

and $\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\beta}\Gamma_{\mu\nu,\beta}$ are

$$\left.\begin{aligned} &\Gamma^{1}_{44} = -\nu' \ e^{2(\nu-\lambda)},\ \Gamma^{4}_{14} = \nu' \\ &\Gamma^{1}_{11} = \lambda',\ \Gamma^{2}_{12} = \Gamma^{3}_{13} = \frac{1}{r},\ \Gamma^{1}_{22} = -re^{-2\lambda} \\ &\Gamma^{3}_{23} = \cot\theta,\ \Gamma^{2}_{33} = -\sin\theta\cos\theta, \\ &\Gamma^{1}_{33} = -re^{-2\lambda}\sin^2\theta \end{aligned}\right\} \qquad (9)$$

where prime denotes derivative w. r. to r.

Since $R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}\left(\Gamma^{\alpha}_{\mu\alpha}\right) - \frac{\partial}{\partial x^{\alpha}}\left(\Gamma^{\alpha}_{\mu\nu}\right) + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu}$

$$= \frac{\partial^2}{\partial x^{\nu}\partial x^{\mu}}\left(\log\sqrt{-g}\right) - \frac{\partial}{\partial x^{\alpha}}\left(\Gamma^{\alpha}_{\mu\nu}\right) + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} - \Gamma^{\beta}_{\mu\nu}\frac{\partial}{\partial x^{\beta}}\left(\log\sqrt{-g}\right) \qquad (10)$$

Also $\sqrt{-g} = r^2\sin\theta e^{\lambda+\nu}$

$$\log\sqrt{-g} = 2\log r + \log\sin\theta + (\lambda + \nu)$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}\left(\log\sqrt{-g}\right)=\frac{2}{r}+\lambda'+\nu' \\
&\frac{\partial^2}{\partial r^2}\left(\log\sqrt{-g}\right)=-\frac{2}{r^2}+\lambda''+\nu'' \\
&\frac{\partial}{\partial \theta}\left(\log\sqrt{-g}\right)=\cot\theta, \\
&\frac{\partial^2}{\partial \theta^2}\left(\log\sqrt{-g}\right)=-\operatorname{cosec}^2\theta
\end{aligned}\right\} \tag{11}$$

$$\frac{\partial}{\partial \phi}\left(\log\sqrt{-g}\right)=0$$

From equation (10) and (11) we get

$$R_{11}=\nu''+\nu'^2-\lambda'\nu'-\frac{2\lambda'}{r} \tag{12}$$

Similarly $R_{22}=e^{-2\lambda}\left(1-r\lambda'+r\nu'\right)-1$ (13)

$$R_{33}=R_{22}\sin^2\theta \tag{14}$$

$$R_{44}=e^{-2\lambda+2\nu}\left[-\nu''+\lambda'\nu'-\nu'^2-\frac{2\nu'}{r}\right] \tag{15}$$

Einstein field equations for empty space are $R_{\mu\nu}=0$

$$\therefore R_{11}=0 \Rightarrow \nu''+\nu'^2-\lambda'\nu'-\frac{2\lambda'}{r}=0 \tag{16}$$

$$R_{22}=0 \Rightarrow e^{-2\lambda}\left(1-r\lambda'+r\nu'\right)-1=0 \tag{17}$$

$$R_{33}=0 \tag{18}$$

$$R_{44}=0 \Rightarrow e^{2(\nu-\lambda)}\left[-\nu''+\lambda'\nu'-\nu'^2-\frac{2\nu'}{r}\right]=0 \tag{19}$$

Equations (16) and (19) give $\lambda'+\nu'=0$

$$\Rightarrow \lambda+\nu=C_1\,(\text{constant}) \tag{20}$$

when $r\to\infty,\ \lambda=\nu=0$ $\quad (20)\Rightarrow C_1=0 \quad \therefore \lambda+\nu=0$

$$(17)\Rightarrow\left(1+2r\nu'\right)e^{2\nu}=1 \quad \Rightarrow\left(re^{2\nu}\right)'=1 \quad \Rightarrow re^{2\nu}=r-2m$$

where m is a constant of integration

$$\therefore g_{44}=e^{2\nu}=1-\frac{2m}{r} \tag{21}$$

Newtonian approximation must hold for weak field i.e. for larger r.
For a special body of mass M

$$\psi = \frac{GM}{r} \quad (22)$$

Since $g_{44} = 1 + \frac{2\psi}{c^2}$

$$= 1 - \frac{2GM}{rc^2} \quad (23)$$

Comparing (21) and (23) we have

$$m = \frac{GM}{c^2}$$

In other wards m represents the mass of the gravitating body

$$\therefore e^{2\nu} = 1 - \frac{2m}{r}$$

$$e^{2\lambda} = e^{-2\nu} = \left(1 - \frac{2m}{r}\right)^{-1}$$

The complete solutions is

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (24)$$

This is the well known Schwarzschild solution valid in empty space outside a spherical object. It holds fairly accurately out side the surface of a star.

Schwarzschild singularity :

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \quad (1)$$

is seen to have the following singularities.

(i) r = 0; the solution becomes singular at r = 0, but this type of singularity also occure in the Newtonian theory.

(ii) r = 2m; the solution again becomes singular at the spherical surface r = 2m. This value of r is known as schwarzschild's radius.

(iii) If we include the cosmological constant Λ then the Schwarzschild solution for empty space corresponding to Einstein field equation.

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Leads to the line element

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (2)$$

Comparing the line element (1) and (2) now see that the effect of the Λ term on the field surrendering on attracti.e point particle increase with the size of the region considered. But the cosmological constant Λ, even if different from zero, is so small that it does not produce any appreciable effects within a region of the order of the solar system.

For entirely empty world, we put $m = 0$, so the Schwarzschild solution (2) becomes

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \left(1 - \frac{\Lambda r^2}{3}\right) dt^2$$

This solution has a singularity at $r = \sqrt{\frac{3}{\Lambda}}$. Since cosmological constant Λ is very small, hence the value of r is very large. It represent the horizon of the world.

4.5. Isotropic co-ordinates:

Let us consider the Schwarzschild exterior solution

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \left(1 - \frac{2m}{r}\right) dt^2$$

Here putting

$$r = \left(1 + \frac{m}{2r_1}\right)^2 r_1 = \left(\frac{4r_1^2 + 4r_1 m + m^2}{4r_1^2}\right) r_1$$

$$= r_1 + m + \frac{m^2}{4r_1}$$

$$dr = dr_1 - \frac{m^2}{4r_1^2} dr_1 = \left(1 - \frac{m^2}{4r_1^2}\right) dr_1$$

$$1 - \frac{2m}{r} = 1 - \frac{2m}{\left(1 + \frac{m}{2r_1}\right)^2 r_1}$$

$$= \frac{\left(1+\frac{m}{2r_1}\right)^2 r_1 - 2m}{\left(1+\frac{m}{2r_1}\right)^2 r_1}$$

$$= \frac{r_1 + m + \frac{m^2}{4r_1^2} - 2m}{\left(1+\frac{m}{2r_1}\right)^2 r_1} = \frac{\left(1-\frac{m}{2r_1}\right)^2}{\left(1+\frac{m}{2r_1}\right)^2}$$

Then equation (1) becomes

$$ds^2 = \frac{-\left(1+\frac{m}{2r_1}\right)^2}{\left(1-\frac{m}{2r_1}\right)^2}\left(1-\frac{m^2}{4r_1^2}\right)dr_1^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \frac{\left(1-\frac{m}{2r_1}\right)^2}{\left(1+\frac{m}{2r_1}\right)^2} dt^2$$

$$= -\left(1+\frac{m}{2r_1}\right)^4 \left[dr_1^2 + \frac{r^2}{\left(1+\frac{m}{2r_1}\right)^2} d\theta^2 + \frac{r_1^2 \sin^2\theta}{\left(1+\frac{m}{2r_1}\right)^2} d\phi^2\right] + \frac{\left(1-\frac{m}{2r_1}\right)^2}{\left(1+\frac{m}{2r_1}\right)^2} dt^2$$

$$= -\left(1+\frac{m}{2r_1}\right)^4 \left[dr_1^2 + r_1^2 d\theta^2 + r_1^2 \sin^2\theta d\phi^2\right] + \frac{\left(1-\frac{m}{2r_1}\right)^2}{\left(1+\frac{m}{2r_1}\right)^2} dt^2 \qquad (2)$$

This is called isotropic line element and the co-ordinates r_1, θ_1, ϕ are called isotropic polar co-ordinates.

Substituting the corresponding Cartesian co-ordinates

$x = r_1 \sin\theta\cos\phi$, $y = r_1 \sin\theta \sin\phi$, $z = r_1 \cos\theta$, we get

$$ds^2 = -\left(1+\frac{m}{2r_1}\right)^4 \left[dx^2 + dy^2 + dz^2\right] + \frac{\left(1-\frac{m}{2r_1}\right)^2}{\left(1+\frac{m}{2r_1}\right)^2} dt^2$$

where $r_1^2 = x^2 + y^2 + z^2$

To obtain the equation of light pulse we put ds = 0

Therefore we get.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \frac{\left(1-\frac{m}{2r_1}\right)^2}{\left(1+\frac{m}{2r_1}\right)^6}$$

Thus the velocity of light distance r_1 from the origin is $\dfrac{1-\frac{m}{2r_1}}{\left(1+\frac{m}{2r_1}\right)^3}$

which is small in all directions.

4.6. Planetary orbit:

Consider the motion of planets in the gravitating field of the sun, where the planets can be regarded as free particle their space time trajectories will be given by the geodesic.

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \qquad (1)$$

Consider the sun as attracting point particle, its gravitational field may be regarded as the field of an isolated particle continually at rest at the origin, so the space time is given by the Schwarzschild line element for empty space viz.

$$ds^2 = -e^{2\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 + e^{2\nu}dt^2$$

where $\lambda = -\nu$ and $e^{2\nu} = e^{-2\lambda} = \left(1-\frac{2m}{r}\right)$

$$x^1 = r,\ x^2 = \theta,\ x^3 = \phi,\ x^4 = t$$

The non vanishing Christoffell's symbols are

$$\left.\begin{array}{c} \Gamma^1_{11} = \lambda',\ \Gamma^1_{44} = \nu' e^{2\nu-2\lambda} \\ \Gamma^1_{22} = -re^{-2\lambda},\ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} \\ \Gamma^1_{33} = -re^{-2\lambda}\sin^2\theta,\ \Gamma^3_{23} = \cot\theta \\ \Gamma^2_{33} = -\sin\theta\cos\theta, \Gamma^4_{14} = \nu' \end{array}\right\} \qquad (3)$$

Taking first θ equation we have from (1) $[\text{ i.e. } \alpha = 2]$

$$\frac{d^2\theta}{ds^2} + \Gamma^2_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

$$\Rightarrow \frac{d^2\theta}{ds^2} + \Gamma^2_{13} \frac{dx^1}{ds} \frac{dx^2}{ds} + \Gamma^2_{21} \frac{dx^2}{ds} \frac{dx^1}{ds} + \Gamma^2_{33} \frac{dx^3}{ds} \frac{dx^3}{ds} = 0$$

$$\Rightarrow \frac{d^2\theta}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \qquad (4)$$

Let us chose the co-ordinate system such that the planet moves initially in a plane $\theta = \pi/2$, so that $\theta = \frac{\pi}{2}$, $\cos\theta = 0$, $\frac{d\theta}{ds} = 0$ (5)

From equations (4) and (5) we get

$$\frac{d^2\theta}{ds^2} = 0 \qquad (6)$$

This equation indicates that the planet continues to move in the plane $\theta = \frac{\pi}{2}$ we shall simplify the remaining geodesic equation with $\theta = \pi/2$.

For r equation i.e. $\alpha = 1$, we have from (1)

$$\frac{d^2x^1}{ds^2} + \Gamma^1_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

$$\Rightarrow \frac{d^2r}{ds^2} + \Gamma^1_{11} \frac{dx^1}{ds} \frac{dx^1}{ds} + \Gamma^1_{22} \frac{dx^2}{ds} \frac{dx^2}{ds} + \Gamma^1_{33} \frac{dx^3}{ds} \frac{dx^3}{ds} + \Gamma^1_{44} \frac{dx^4}{ds} \frac{dx^4}{ds} = 0$$

$$\Rightarrow \frac{d^2r}{ds^2} + \lambda'\left(\frac{dr}{ds}\right)^2 - re^{-2\lambda}\left(\frac{d\phi}{ds}\right)^2 + \nu' e^{2\nu-2\lambda}\left(\frac{dt}{ds}\right)^2 = 0 \qquad (7)$$

For the ϕ equation we have from (1) $[i.e. \alpha = 3]$

$$\frac{d^2x^3}{ds^2} + \Gamma^3_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

$$\Rightarrow \frac{d^2\phi}{ds^2} + \Gamma^3_{23} \frac{dx^2}{ds} \frac{dx^3}{ds} + \Gamma^3_{32} \frac{dx^3}{ds} \frac{dx^2}{ds} + 2\,\Gamma^3_{13} \frac{dx^1}{ds} \frac{dx^3}{ds} = 0$$

$$\Rightarrow \frac{d^2\phi}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\phi}{ds} = 0 \quad \left[\because \frac{d\theta}{ds} = 0\right] \qquad (8)$$

Finally for t equation we have from (1) $(i.e. \alpha = 4)$

$$\frac{d^2x^4}{ds^2} + \Gamma^4_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

$$\Rightarrow \frac{d^2t}{ds^2} + 2\Gamma_{14}^4 \; \frac{dx^1}{ds} \; \frac{dx^4}{ds} = 0$$

or, $$\frac{d^2t}{ds^2} + 2\nu' \frac{dr}{ds} \; \frac{dt}{ds} = 0 \qquad (9)$$

Equation (8) gives

$$\frac{d}{ds}\left(r^2 \frac{d\phi}{ds}\right) = 0$$

Integrating

$$r^2 \frac{d\phi}{ds} = h \qquad (10)$$

where h is the constant of integration. Equation (9) gives

$$\frac{d}{ds}\left(e^{2\nu} \frac{dt}{ds}\right) = 0$$

Integrating

$$e^{2\nu} \frac{dt}{ds} = k$$

where k is the constant of integration

$$\Rightarrow \frac{dt}{ds} = ke^{-2\nu} \qquad (11)$$

We have from (2)

$$ds^2 = -e^{2\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 + e^{2\nu}\left(\frac{dt}{ds}\right)^2$$

$$\Rightarrow 1 = -e^{2\lambda}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 - r^2 \sin^2\theta\left(\frac{d\phi}{ds}\right)^2 + e^{2\nu}\left(\frac{dt}{ds}\right)^2$$

Putting $\theta = \frac{\pi}{2}, \; \frac{d\theta}{ds} = 0$

$$e^{2\lambda}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - e^{2\nu}\left(\frac{dt}{ds}\right)^2 + 1 = 0$$

$$\Rightarrow e^{2\lambda}\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2} - k^2 e^{-2\nu} + 1 = 0 \qquad \text{[using (10) and (11)]}$$

Putting $r = \frac{1}{u}$

$$\frac{dr}{ds} = \frac{dr}{du}\frac{du}{ds} = -\frac{1}{u^2}\frac{du}{ds}$$

$$\Rightarrow \frac{1}{u^4}\left(\frac{du}{ds}\right)^2 + h^2u^2e^{-2\lambda} - k^2e^{-2\nu-2\lambda} + e^{-2\lambda} = 0$$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 + u^2e^{2\nu} - \frac{k^2}{h^2} + \frac{e^{-2\nu}}{h^2} = 0 \qquad [\lambda = -\nu]$$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 + u^2 - 2mu^3 - \frac{2mu}{h^2} + \left(\frac{1-k^2}{h^2}\right) = 0$$

Differentiating w.r to ϕ

$$2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + 2u\frac{du}{d\phi} - 6mu^2\frac{du}{d\phi} - \frac{2m}{h^2}\frac{du}{d\phi} = 0$$

Dividing by $2\frac{du}{d\phi}$

$$\Rightarrow \frac{d^2u}{d\phi^2} + u = 3mu^2 + \frac{m}{h^2} \qquad (12)$$

This is the required differential equation of the path of a planet, comparing this with the Newtonian orbit namely.

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2}$$

We find that $3mu^2$ is an additional term in equation (12). This term is defined as the relativistic correction of Newtonian orbit.

The ratio of $= \dfrac{3mu^2}{\frac{m}{h^2}} = 3h^2u^2$

$$= 3\left(r^2\frac{d\phi}{ds}\right)^2\frac{1}{r^2}$$

$$= 3r^2\left(\frac{d\phi}{ds}\right)^2 = 3\left(r\frac{d\phi}{ds}\right)^2 = 3\left(\frac{\textit{transveral velocity of planet}}{\textit{Velocity of light}}\right)^2$$

For ordinary speeds this ratio is extremely small. For example in the case of the motion of the earth around the sun, this ratio comes out to be 3×10^{-8}. Hence for ordinary speed the relativistic effects are negligible and therefore the extra term $3mu^2$ in equation (12) in practical cases represents an in appreciable correction to the Newtonian orbit. Consequently the difference between the relativistic and the Newtonian theory of gravitation is only slight.

4.7. Crucial tests in relativity:

There are three important consequences of Einstein's theory of gravitation, which has been experimentally verified and are known as the experimental or critical tests in the theory of relativity. Moreover they provided a means to compare the relativistic theory with Newtonian theory.

There are the following :

(i) The advance of perihelion of the planets.

(ii) The bending of light rays in a gravitational field.

(iii) The gravitational red shift of speetral lines.

(iv) The advance of perihelion of the planets:

The relativistic differential equation of the path of the planet is

$$\frac{d^2u}{d\phi^2}+u=\frac{m}{h^2}+3mu^2,\ u=\frac{1}{r} \tag{1}$$

with $r^2\frac{d\phi}{ds}=h$

As a first approximation, the small term $3mu^2$ can be neglected, so that we have

$$\frac{d^2u}{d\phi^2}+u=\frac{m}{h^2} \tag{2}$$

The solution of (2) is

$$u=\frac{m}{h^2}\left[1+e\cos(\phi-\omega)\right] \tag{3}$$

where e and ω are constants, e being the eccentricity of the orbit and ω the initial longitude of the perihelion.

To obtain the second approximation, substituting (3) on the second term on R.H.S. of equation (1) we get

$$\frac{d^2u}{d\phi^2}+u=\frac{m}{h^2}+3m\frac{m^2}{h^4}\left[1+e^2\cos^2(\phi-\omega)+2e\cos(\phi-\omega)\right]$$

$$=\frac{m}{h^2}+\frac{3m^3}{h^4}+\frac{6m^3}{h^4}e\ \cos(\phi-\omega)+\frac{3}{2}\ \frac{m^3e^2}{h^4}\left(1+\cos^2(\phi-\omega)\right)$$

The above equation the term which can produce any effect within the range of observation is the term $\cos(\phi-\omega)$.

Now we know that the particular integral of the equation

$$\frac{d^2u}{d\phi^2}+u=A\cos\phi$$

$$i.e.\ u_1=\frac{1}{2}\ A\phi\ \sin\phi$$

The additional term $\frac{6m^3e}{h^4}\cos(\phi-\omega)$ in the R.H.S. of (4) gives a part of u given by

$$u_1=\frac{3m^3e}{h^4}\phi\sin(\phi-\omega)$$

Hence the solution of (1) of the second order of approximation is

$$u=\frac{m}{h^2}\left[1+e\cos(\phi-\omega)\right]+u_1$$

$$u=\frac{m}{h^2}\left[1+e\cos\ \ (\phi-\omega)\right]+\frac{3m^3}{h^4}e\phi\sin(\phi-\omega)$$

$$=\frac{m}{h^2}+\frac{me}{h^2}\left[\cos(\phi-\omega)+\frac{3m^2}{h^2}\phi\sin(\phi-\omega)\right]$$

Substituting

$$\frac{3m^2\phi}{h^2}=\delta\omega \tag{6}$$

Since $\delta\omega$ is very small so that we may write $\sin\delta\omega=\delta\omega$ and $\cos\delta\omega=1$

$$\therefore u=\frac{m}{h^2}+\frac{me}{h^2}\left[\cos(\phi-\omega)\cos\delta\omega+\sin\delta\omega\sin(\phi-\omega)\right]$$

$$=\frac{m}{h^2}\left[1+e(\phi-\omega-\delta\omega)\right] \tag{7}$$

This equation represents the solution of (1) for the orbit of the planet to the second order of approximation. In this equation the term δw corresponds to an advance in perihelion of planet. Accordingly when a planet moves round the sun through one revolution viz $\phi=2\pi$ the perihelion of the planet does not return to its initial position, but advance by an angle,

$$\delta\omega = \frac{3m^2}{h^2} \times 2\pi = \frac{6\pi m^2}{h^2} \quad (8)$$

Using the standard relation

$$\frac{m}{h^2} = \frac{1}{l},$$

where l is the semi latus rectum of the orbit, we have

$$h^2 = ml = ma\ \left(1-e^2\right)$$

Also from keplar's 3rd law, the time period T of the planet is

$$T = \frac{2\pi}{\sqrt{m}}\ a^{\frac{3}{2}} \quad (9)$$

'a' being the semi major axis of the orbit

From (8) and (9) we have

$$\delta\omega = \frac{24\pi^3 a^2}{T^2\left(1-e^2\right)}$$

Restoring the velocity of light c, we get

$$\delta\omega = \frac{24\pi^3 a^2}{c^2 T^2\left(1-e^2\right)} \quad (10)$$

For mercury $e = 0.2056,\ a = 0.6 \times 10^8\,km\ \ c = 3 \times 10^8\ {}^{m}\!/_{\text{sec}}$ and T = 88 days. The value of δw per century for mercury has been found from (10) to be 42.9 sec. and the observed value 43.5 secs.

It thus follows that the shift of the perihelion of Mercury as predicted by general theory of relativity agrees with the observed tests for general relativity.

(ii) The bending of light says in a gravitational field :

We consider the deflection of a light ray in the gravitational field of the sun. According to the general theory of relativity the track of a light ray is given by geodesic equations with the added condition $ds = 0$. It means that the differential equation of planetary orbit viz.

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2 \quad (1)$$

with $r^2 \dfrac{d\phi}{ds} = h$ (2)

is also applicable to the path of a light ray.

For the track of a light ray $ds = 0$ so that the equation (2) yields $h \to \infty$. Hence the track of a light ray in the neighborhood of a gravitating mass m is given by

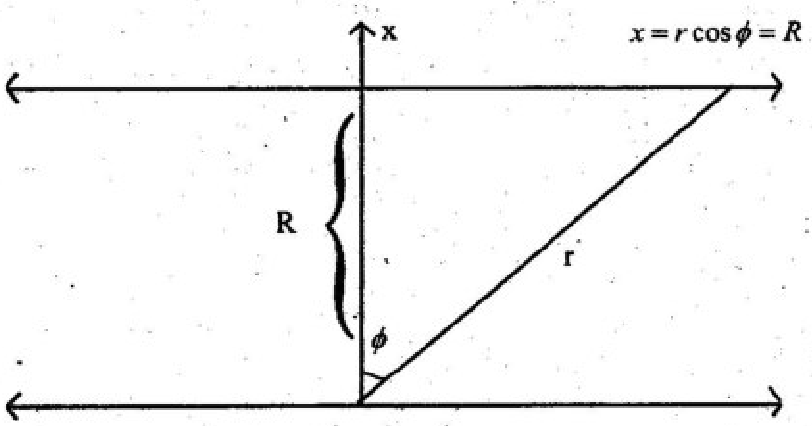
$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \qquad (3)$$

Neglecting the small term $3mu^2$ to the first approximation we get

$$\frac{d^2u}{d\phi^2} + u = 0 \qquad (4)$$

Its solution is

$$u = A\cos\phi + B\sin\phi \qquad (5)$$



where A and B are constants.

$$(5) \Rightarrow \frac{du}{d\phi} = -A\sin\phi + B\cos\phi \qquad (6)$$

The boundary condition are

$$\phi = 0, \quad u = \frac{1}{R} \text{ and } \frac{du}{d\phi} = 0$$

$$(5) \Rightarrow \frac{1}{R} = A, \qquad (6) \Rightarrow B = 0$$

$$\therefore u = \frac{1}{R}\cos A \qquad (7)$$

To obtain the second approximation let us substitute this value of u and the R.H.S. of (3) we have

$$\frac{d^2u}{d\phi^2} + u = \frac{3m}{R^2}\cos^2\phi \qquad (8)$$

The particular solution u_1 of (8) is

$$u_1 = \frac{1}{D^2+1}\left(\frac{3m}{R^2}\cos^2\phi\right)$$

$$= \frac{3m}{2R^2}\frac{1}{D^2+1}(1+\cos 2\phi)$$

$$= \frac{3m}{2R^2}\left(1+\frac{\cos 2\phi}{1-2^2}\right)$$

$$= \frac{3m}{2R^2}\left(1-\frac{\cos 2\phi}{3}\right)$$

$$= \frac{m}{2R^2}\left(3\cos^2\phi + 3\sin^2\phi - \cos^2\phi + \sin^2\phi\right)$$

$$= \frac{m}{2R^2}\times 2\left(\cos^2\phi + 2\sin^2\phi\right) = \frac{m}{R^2}\left(\cos^2\phi + 2\sin^2\phi\right)$$

Hence the complete solution of equation (3) to the second approximation is

$$u = \frac{1}{R}\cos\phi + \frac{m}{R^2}\left(\cos^2\phi + 2\sin^2\phi\right)$$

$$\Rightarrow R = r\cos\phi + \frac{m}{R}\left(r\cos^2\phi + 2r\sin^2\phi\right)$$

Introducing the Cartesian co-ordinates, we have,

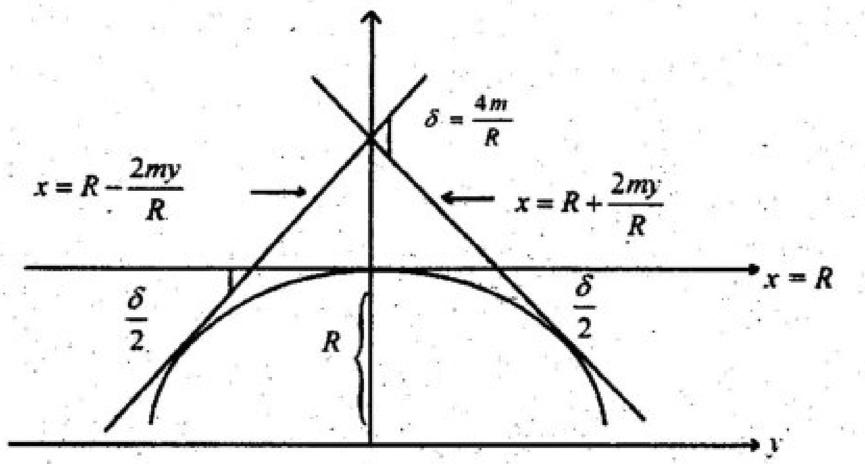
$x = r\cos\phi,\ y = r\sin\phi$

$$R = x + \frac{m}{R}\frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

$$\Rightarrow x = R - \frac{m}{R}\frac{\left(x^2+2y^2\right)}{\sqrt{x^2+y^2}} \qquad (9)$$

In this equation the second term measures the very slight deviation from straight path $x = R$. The asymptotes to (9) can be found by taking y very large as compared to x, so the equations of the asymptotes to curve (9) are $x = R - \frac{m}{R}(\pm 2y)$

$$\left.\begin{aligned} &\Rightarrow x = R + \frac{2my}{R} \\ &and\ \ x = R - \frac{2my}{R} \end{aligned}\right\} \qquad (10)$$



If δ is the angle between the asymptotes, we have

$$\tan\delta = \frac{\frac{R}{2m} - \left(-\frac{R}{2m}\right)}{1 + \frac{R}{2m}\left(-\frac{R}{2m}\right)}$$

$$= \frac{4mR}{4m^2 - R^2}$$

$$\sin\delta = \frac{4mR}{4m^2 + R^2}$$

Since $4m^2 << R^2$ and hence neglected.

$$\sin\delta = \frac{4mR}{R^2} = \frac{4m}{R}$$

Since δ is very small, $\sin\delta \to \delta$

$$\therefore \delta = \frac{4m}{R} = \frac{4GM}{c^2 R}$$

This equation represents the total deflection of a light ray passing near a heavy mass M.

For a light ray grazing the surface of the sun.

$M = 1.92 \times 10^{30} kg$, $R = 6.97 \times 10^8 m$

$c = 3 \times 10^8 m/sec$ $G = 6.66 \times 10^{-11} m^3 kg^{-1}.sec^{-1}$

$\therefore \delta = 8.36 \times 10^6$ radian

$= 1.75$ seconds.

This implies that a light ray grazing in the surface of the sun will be deflected by an angle 1.75 seconds of arc, which is the amount observed by astronomers during the many total eclipses of the sun, since the year 1919, when the first measurement of the deflection of light was made. Thus the theoretical prediction of the bending of light rays in the gravitational field agrees with the observations.

(iv) The gravitational red shift of spectral lines:

Making use of schwarzscild line element, let us now investigate whether Einstein theory of relativity accounts numerically for the observed red shift of spectral lines emitted by an atom should in a gravitational field when this light is observed on the surface of the earth. Consider a number of similar atoms vibrating at different points in the region. Consider an atom to be momentarily at rest in co-ordinate system$(r,\ \theta,\ \phi,\ t)$

The Schwarschild line element due to a gravitating mass m is

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \left(1-\frac{2m}{r}\right) dt^2 \tag{1}$$

Where $r,\ \theta,\ \phi,\ t$ are co-ordinates as observed by the distant observers.

For an atom (stationary) on the sun emitting light

$dr = d\theta = d\phi = 0$ so that (1) takes the form

$$ds^2 = \left(1-\frac{2m}{r}\right) dt^2 \tag{2}$$

$$\Rightarrow ds = dt\left(1-\frac{m}{r}\right) \tag{3}$$

Since $\frac{m}{r}$ is very small

It $dt = T$ (period of light) as observed by a distant observer then ds = T_0 (period of light) as observed as the sun, which is the proper period of an atom. Hence from (3)

$$T_0 = T\left(1-\frac{m}{r}\right)$$

which implies that a periodic phenomena in the gravitational field of a heavy mass will appear in an observer out side the field as slowed down.

$$\therefore T = T_0\left(1+\frac{m}{r}\right) \quad \left[\because \frac{m}{r} \text{ is very small}\right]$$

$$\Rightarrow \frac{1}{\nu} = \frac{1}{\nu_0}\left(1+\frac{m}{r}\right)$$

$$\Rightarrow \frac{c}{v} = \frac{c}{v_0}\left(1+\frac{m}{r}\right)$$

$$\Rightarrow \lambda = \lambda_0\left(1+\frac{m}{r}\right)$$

$$\Rightarrow \lambda - \lambda_0 = \frac{m}{r}\lambda_0$$

$$\Rightarrow \frac{\delta\lambda}{\lambda_0} = \frac{m}{r} \qquad (4)$$

i.e. when r is increasing change in wave length is decreasing that means wave length is increasing and hence frequency is decreasing in red shifted.

Thus the wave increases as it leaves the gravitational field. This means that there is a shift toward the red end of the spectrum.

$$\therefore \frac{\delta\lambda}{\lambda_0} = \frac{GM}{c^2 r}$$

For the gravitational field of the sun this shift is qualitafively observable and the amount agrees with equation within the error of observation.

4.8. Schwarzschild's interior solution:

We shall now determine line element inside the sphere of matter. It is natural that such a solution must depend of the properties of fluid of which the sphere is composed. Schwarzschild solved this problem by assuming that the sphere is composed of an incompressible perfect fluid of proper density ρ_0. The solution of these equations must satisfy the following boundary conditions:

(i) The pressure is zero at the boundary of the sphere.

(ii) The density ρ_0 is uniform through out the sphere.

We take the line element of the from

$$ds^2 = -e^{2\lambda}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 + e^{2\nu}dt^2 \qquad (1)$$

where λ and ν are functions of r only

We have,

$$T^{ij} = (\rho_0 + p_0)\frac{dx^i}{ds}\frac{dx^j}{ds} - g^{ij}p_0 \qquad (2)$$

Since the distribution of mass is static, all velocity components of the fluid matter must be zero i.e.

$$\frac{dx^1}{ds}=\frac{dr}{ds}=0,\ \ \frac{dx^2}{ds}=\frac{d\theta}{ds}=0,\ \ \frac{dx^3}{ds}=\frac{d\phi}{ds}=0$$

So from equation (1) we have

$$\frac{dx^4}{ds}=\frac{dt}{ds}=e^{-\nu} \tag{3}$$

$$(2)\Rightarrow T_1^1=(\rho_0+p_0)\ \frac{dx^1}{ds}\ \frac{dx^1}{ds}\ -\delta_1^1 p_0\ \ =-p_0$$

Similarly we can show

$$T_2^2=T_3^3=-p_0$$

Also, $$T_4^4=(\rho_0+p_0)\ \frac{dx^4}{ds}\ \frac{dx^4}{ds}\ -\ \delta_4^4 p_0\ \ =\rho_0$$

$$\therefore T_1^1=T_2^2=T_3^3=-p_0,\ T_4^4=\rho_0 \tag{4}$$

Einstein field equation for the presence of matter are

$$R_j^i-\frac{1}{2}\delta_j^i\ \ R+\Lambda\delta_j^i=-8\pi\ \ T_j^i$$

$$\Rightarrow -8\pi T_j^i=g^{i\alpha}R_{j\alpha}-\frac{1}{2}\delta_j^i\ \ R+\Lambda\delta_j^i \tag{5}$$

The non vanishing components of contracted curvature tensor are the following

$$\left.\begin{aligned}
R_{11}&=\nu''-\lambda'\nu'+\nu'^2-\frac{2\lambda'}{r}\\
R_{22}&=(1-r\lambda'+r\nu')e^{-2\lambda}-1\\
R_{33}&=R_{22}\sin\theta\\
R_{44}&=\left(-\nu''+\lambda'\nu'-\nu'^2-\frac{2\lambda'}{r}\right)e^{2\nu-2\lambda}
\end{aligned}\right\} \tag{6}$$

where prime denote differentiation w.r.t r

From (4) and (5) we have

$$8\pi p_0=g^{11}R_{11}-\frac{1}{2}R+\Lambda \tag{7}$$

$$8\pi p_0=g^{22}R_{22}-\frac{1}{2}R+\Lambda \tag{8}$$

$$8\pi p_0=g^{33}R_{33}-\frac{1}{2}R+\Lambda \tag{9}$$

$$-8\pi\rho_0=g^{44}R_{44}-\frac{1}{2}R+\Lambda \tag{10}$$

Equation (9) can be expressed as

$$8\pi p_0 = g^{33}R_{33} - \frac{1}{2}R + \Lambda \tag{11}$$

$$= -\frac{1}{r^2\sin^2\theta}R_{33} - \frac{1}{2}R + \Lambda$$

$$= -\frac{1}{r^2}R_{22} - \frac{1}{2}R + \Lambda$$

$$= g^{22}R_{22} - \frac{1}{2}R + \Lambda$$

It means (8) and (9) are indentical

Also $R = g^{ij}R_{ij}$

$$= g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44}$$

Using (6) we have

$$R = -2e^{-2\lambda}\left(\nu'' + \nu'^2 + \lambda'\nu' - \frac{2\lambda'}{r} + \frac{2\nu'}{r}\right) - \frac{1}{r^2} \tag{11}$$

(6), (7), (11) $\Rightarrow$

$$8\pi p_0 = e^{-2\lambda}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \tag{12}$$

(6), (8), (10) $\Rightarrow$

$$8\pi p_0 = e^{-2\lambda}\left(\nu'' - \lambda'\nu' + \nu'^2 - \frac{\lambda'}{r} + \frac{\nu'}{r}\right) + \Lambda \tag{13}$$

(6), (10) and (11) $\Rightarrow$

$$8\pi\rho_0 = e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \tag{14}$$

(12)+(14) $\Rightarrow$

$$8\pi(\rho_0 + p_0) = e^{-2\lambda}\left(\frac{2\lambda'}{r} + \frac{2\nu'}{r}\right) \tag{15}$$

$$\Rightarrow 8\pi(\rho_0 + p_0)\nu' = e^{-2\lambda}\left(\frac{2\lambda'\nu'}{r} + \frac{2\nu'^2}{r}\right) \tag{16}$$

From (12) we have

$$8\pi\frac{dp_0}{dr} = e^{-2\lambda}(-2\lambda')\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) + e^{-2\lambda}\left(\frac{2\nu''}{r} - \frac{2\nu'}{r^2} - \frac{2}{r^3}\right) + \frac{2}{r^3} \tag{17}$$

From (16) and (17) we have

$$8\pi\left[\frac{dp_0}{dr}+(\rho_0+p_0)\nu'\right]=e^{-2\lambda}\left[\frac{2\nu''}{r}-\frac{2\lambda'\nu'}{r}-\frac{2}{r^3}+\frac{2\nu'}{r}-\frac{2\lambda'+2\nu'}{r^2}\right]+\frac{2}{r^3} \qquad (18)$$

From (12) and (13) we have

$$e^{-2\lambda}\left[\frac{2\nu'}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2}=e^{-2\lambda}\left(\nu''-\lambda'\nu'+\nu'^2-\frac{\lambda'}{r}+\frac{\nu'}{r}\right)$$

$$\Rightarrow e^{-2\lambda}\left(\nu''-\lambda'\nu'+\nu'^2-\frac{\lambda'}{r}+\frac{\nu'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0 \qquad (19)$$

Using (19) in (18) we have

$$8\pi\left[\frac{dp_0}{dr}+(\rho_0+p_0)\nu'\right]=0$$

$$i.e.\ \frac{dp_0}{dr}+(p_0+\rho_0)\nu'=0 \qquad (20)$$

From (14) we have,

$$(8\pi\rho_0+\Lambda)r^2=e^{-2\lambda}(2r\lambda'-1)+1$$

$$\Rightarrow \frac{d}{dr}\left(re^{-2\lambda}\right)=1-(8\pi\rho_0+\Lambda)r^2$$

Integrating we get

$$re^{-2\lambda}=r-\frac{8\pi\rho_0+\Lambda}{3}r^3+C_1$$

$$\Rightarrow e^{-2\lambda}=1-\frac{8\pi\rho_0+\Lambda}{3}+\frac{C_1}{r}$$

Taking

$$\frac{1}{R^2}=\frac{8\pi\rho_0+\Lambda}{3} \quad \text{we get}$$

$$e^{-2\lambda}=1-\frac{r^2}{R^2}+\frac{C_1}{r}$$

In order to remove the singularity at $r = 0$ take $C_1 = 0$

$$\therefore e^{-2\lambda}=1-\frac{r^2}{R^2}$$

From (20) we have

$$\frac{dp_0}{p_0+\rho_0}=-d\nu$$

$$\Rightarrow p_0+\rho_0=C_2e^{-\nu}$$

$$\Rightarrow 8\pi\left(p_0+\rho_0\right)=8\pi C_2 e^{-\nu} \;=C_3 e^{-\nu}$$

$$\Rightarrow 8\pi e^{\nu}\left(p_0+\rho_0\right)=C_3$$

Using (16) we have

$$2e^{\nu}\frac{\lambda'+\nu'}{r}e^{-2\lambda}=C_3$$

But

$$\frac{2\left(\lambda'+\nu'\right)}{r}e^{-2\lambda}=e^{-2\lambda}\frac{2\lambda'}{r}+e^{-2\lambda}\frac{2\nu'}{r}$$

Since $\quad e^{-2\lambda}=1-\dfrac{r^2}{R^2}$

$$\Rightarrow e^{-2\lambda}\left(-2\lambda'\right)=-\frac{2r}{R^2}$$

$$\therefore \frac{2\left(\lambda'+\nu'\right)}{r}e^{-2\lambda}=\frac{2}{R^2}+e^{-2\lambda}\frac{2\nu'}{r}$$

$$=\frac{2}{R^2}+\frac{2\nu'}{r}\left(1-\frac{r^2}{R^2}\right)$$

$$\therefore e^{\nu}\left[\frac{2}{R^2}+\frac{2\nu'}{r}\left(1-\frac{r^2}{R^2}\right)\right]=C_3$$

Putting $e^{\nu}=u\Rightarrow \dfrac{du}{dr}=e^{\nu}\nu'$

$$\therefore \frac{2u}{R^2}+\frac{2}{r}\;\frac{du}{dr}\left(1-\frac{r^2}{R^2}\right)=C_3$$

$$\Rightarrow \frac{du}{dr}+\frac{r}{R^2-r^2}u=\frac{C_3}{2}R^2\frac{r}{R^2-r^2}$$

$$\Rightarrow \frac{du}{dr}+\frac{r}{R^2-r^2}u=\frac{C_4 r}{R^2-r^2} \qquad (21)$$

which is of the form

$$\frac{du}{dr}+P(r)u=Q(r)$$

Hence its solution is

$$\frac{u}{\sqrt{R^2-r^2}}=\frac{C_4}{\sqrt{R^2-r^2}}+C_5$$

$$\Rightarrow u=C_4+C_5\sqrt{R^2-r^2}$$

$$\Rightarrow e^{\nu} = C_4 + C_5\sqrt{R^2 - r^2}$$

$$= A - B\sqrt{1 - \frac{r^2}{R^2}}$$

where $C_5 R = -B$, $C_4 = A$ and $\frac{1}{R^2} = \frac{8\pi\rho_0 + \Lambda}{3}$

From (12) we have

$$8\pi p_0 = e^{-2\lambda}\frac{2\nu'}{r} + \frac{e^{-2\lambda}}{r^2} - \frac{1}{r^2} + \Lambda$$

$$= e^{-2\lambda}\frac{2\nu'}{r} + \frac{1}{r^2} - \frac{1}{R^2} + \Lambda - \frac{1}{r^2}$$

$$= e^{-2\lambda}\frac{2\nu'}{r} - \frac{1}{R^2} + \Lambda \tag{22}$$

$$e^{\nu}\nu' = -B \times \frac{1}{2}\frac{-2 \times \frac{r}{R^2}}{\sqrt{1 - \frac{r^2}{R^2}}} = B\frac{\frac{r}{R^2}}{\sqrt{1 - \frac{r^2}{R^2}}}$$

$$\frac{\nu'}{r} = \frac{\frac{B}{R^2}}{\sqrt{1 - \frac{r^2}{R^2}}} \times \frac{1}{A - B\sqrt{1 - \frac{r^2}{R^2}}}$$

$$e^{-2\lambda}\frac{\nu'}{r} = \frac{\frac{B}{R^2}\left(1 - \frac{r^2}{R^2}\right)}{\sqrt{1 - \frac{r^2}{R^2}}\left(A - B\sqrt{1 - \frac{r^2}{R^2}}\right)}$$

$$\Rightarrow e^{-2\lambda}\frac{2\nu'}{r} - \frac{1}{R^2} = \frac{3B\sqrt{1 - \frac{r^2}{R^2}} - A}{R^2\left(A - B\sqrt{1 - \frac{r^2}{R^2}}\right)}$$

using (22) we have,

$$8\pi p_0 = \frac{3B\sqrt{1-\frac{r^2}{R^2}} - A}{R^2\left(A-B\sqrt{1-\frac{r^2}{R^2}}\right)} + \Lambda \tag{23}$$

The Λ is an important factor in case the distance from the origin is very large in comparison to r_1 (The radius of the massive body).

Hence we take $\Lambda = 0$ for $r \le r_1$

Also $p_0 = 0$ for $r = r_1$

$\therefore p_0 = \Lambda = 0$ for $r = r_1$

using this result in (23) we get

$$A = 3B\sqrt{1-\frac{r^2}{R^2}} \tag{24}$$

The line element for an interval in the interior of the massive body

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 - r^2 d\theta - r^2 \sin^2\theta d\phi^2 + \left(A - B\sqrt{1-\frac{r^2}{R^2}}\right)^2 dt^2$$

Which is the schwarzschild's interior solution.

The exterior solution of the same body

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \left(1-\frac{2m}{r}\right) dt^2$$

The exterior and interior solutions becomes identical at $r = r_1$

$$1-\frac{r_1^2}{R^2} = 1-\frac{2m}{r_1} = \left(A - B\sqrt{1-\frac{r_1^2}{R^2}}\right)$$

$$\Rightarrow 1-\frac{r_1^2}{R^2} = 1-\frac{2m}{r_1} = 4B^2\left(1-\frac{r_1^2}{R^2}\right) \quad \text{[using (24)]}$$

$$\Rightarrow 1-\frac{r_1^2}{R^2} = 1-\frac{2m}{r_1},\ 1-\frac{r_1^2}{R^2} = 4B^2\left(1-\frac{r_1^2}{R^2}\right)$$

$$\Rightarrow m = \frac{r_1^3}{2R^2},\ 4B^2 = 1$$

$$\Rightarrow \frac{4}{3}\pi r_1^3 \rho_0 = \frac{r_1^3}{2R^2},\ 2B = 1$$

$$\Rightarrow \frac{8\pi\rho_0}{3} = \frac{1}{R^2},\ B = \frac{1}{2}$$

It is accordance with the equation $\frac{1}{R^2} = \frac{8\pi\rho_0 + \Lambda}{3}$ where $\Lambda = 0,\ r = r_1$

From (24) we have

$$A = \frac{3}{2}\left(1 - \frac{r_1^2}{R^2}\right)^{\frac{1}{2}},\ B = \frac{1}{2}$$

$$\therefore \frac{1}{R^2} = \frac{8\pi\rho_0}{3}$$

The interior solution is real iff

$$\frac{2m}{r} < 1 \text{ i.e. } \frac{r_1^2}{R^2} < 1$$

$$\Rightarrow r_1^2 < R^2 = \frac{3}{8\pi\rho_0}$$

Exercise

1. Write a short note on the energy momentum tensor $T^{\mu\nu}$ and discuss the reasons which led Einstein to chose the field equations in the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu}$$

Show further that these equation reduce in linear approximation to Newtonian equations.

$$\nabla^2 \psi = -8\pi\rho$$

2. Obtain Schwarzschild exterior solution for the gravitational field of a single mass at rest and explain on the basis of this solution the advance of perihelion of the planet Mercury.
3. Discuss the three crucial tests of general relativity and the support they led to the theory.

4. For the line element.

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \left(1-\frac{2m}{r}\right) dt^2$$

where m is a constant

Write down the differential equation of the geodesics and show that in the plane $\theta = \frac{\pi}{2}$

these equation reduce to $\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2,\ u = \frac{1}{r},\ r^2\frac{d\phi}{ds} = h$

Solve these equations and discuss any one of the crucial test with the help of your solution.

5. Derive the equation of planetary orbit in the general theory of relativity.

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Unit 5

Cosmology

5.0. Introduction:-

The speculation about the nature of the universe are as old as the man himself. However, it goes to the credit of the General Theory of Relativity that for the first time in the history of physical sciences, it provide a mathematical frame work to explore the nature of the universe and shifted the cosmological problem from a purely speculative plane to a quantitative one. It occurred in 1917, when Einstein paper "Cosmological Consideration in General Relativity" appeared in Prussian Academy of science.

The three crucial tests of general theory of relativity do indicate that it has provided some significant advance and improvement over the Newtonian theory and has furnished an accepted solution of the problem of the field of a star in the empty space surrounding it at least to distance of the order of the dimensions to extended its applications to regions beyond the safer system, viz. to the universe as a whole. The justification for such an application of three gravitational theory to explain the structure of the universe can be near in the observation that there is a general tendency among the stars to cluster together in the form of a nebulac, which themselves again show a similar tendency to cluster to some extended.

This indicates that there does not exists some sort of gravitational action extending upto vast distance of the universe, a phenomenon which would be predicted by the relativistic theory of gravitation.

To consider the universe as a whole, certain simplifying assumptions to idealize the universe have to be made as is done for example by geodesist, who speaking from a large scale point approximates the shape of the earth by an ellipsoid through on a smaller scale it represents much complicated picture.

As a first step in this direction, one should naturally ignore all local concentration of matter, smooth out all irregularities and assume the universe to be filled with a uniform proper density ρ_0 of the matter fluid. At the time when Einstein investigated the problems no large scale motion of matter was known to exists and he was led to ignore all small disturbances. Thus according to Einstein, the matter should be chosen in proper co-ordinate system and the proper distances of the nebulac from the observer should not alter with time.

There are two more assumptions implicitly contained in Einstein's original pare, these are

(i) Isotropy of the universe,

(ii) Homogeneity of universe.

By isotropy we mean that the spatial view of the universe is rotationally invariant about a point in space i.e. all spatial directions are equivalent. Homogeneity means that the history of the universe is invariant under spatial translations. In other words it is impossible to distinguish one plane in the universe from the other.

Thus Einstein's original investigation are based on the following assumption:

(i) The universe is static, i.e. in a proper co-ordinate system matter at rest and the proper pressure p_0 and proper density ρ_0 are the same every where.

(ii) The universe is isotropic i.e. all spatial directions are equivalent.

(iii) The universe is homogeneous i.e. no part of the universe can be distinguished from any other.

It is interesting at this stages to ask the question "are these assumptions consistant"? i.e. is it possible to satisfy the postulates of the isotropy and homogeneity and of resting masses in the new theory of gravitation viz Einstein field equation.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi\, GT_{\mu\nu} \qquad (1)$$

It will be shown later a straight forward detailed calculation shows that the three postulates are not compatible with Einstein's original equations given by (1) Einstein, however, found that a modification of his original field equations of general relativity by introducing an additional small constant term Λ which is called cosmological constant, does render the general theory of relativity consistent with the above postulates. The Einstein's modified filed equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^2} T_{\mu\nu}$$

The constant Λ is such that its effect is negligible for phenomena in the solar system or even in our own galaxy, but becomes important when the universe as a whole is considered.

5.1 static cosmological models : The models of the universe based on the above three assumptions lead to what are called "Static Cosmological Models". The word "static' is applied in view of the first assumption viz the static nature of the matter distribution. The following three line elements

(i) Einstein line element.

(ii) De-Sitter line element.

(iii) Special relativity line element exhaust all possibilities which are admitted by a static, isotropic and homogeneous universe.

The line element of a static, homogenous and isotropic universe has the familiar form:

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta \, d\phi^2 \qquad (1)$$

where (r, θ, ϕ) are the spherical polar co-ordinates and

$$\nu = \nu(r), \qquad \lambda = \lambda(r) \qquad (2)$$

are some unknown functions of the radial distance r to be determined.

The pressure p_0 and density ρ_0 determined by field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \qquad (3)$$

where G = c = 1 are given by

$$\left.\begin{aligned} 8\pi p_0 &= e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \\ 8\pi \rho_0 &= e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \\ \frac{dp_0}{dr} &= -\frac{1}{2}(p_0 + \rho_0)\nu' \end{aligned}\right\} \qquad (4)$$

where a prime denotes derivatives w-r to r i.e. $\nu' = \frac{d\nu}{dr}$ etc (5)

The solution of the above equation must be compatible with the postulates made earlier and hence the following requirements must be satisfied.

(a) The pressure p_0 the density ρ_0 as measured by a local observer should every where have the same constant value.

(b) For small values of r the line element should reduce to the flat space time of special relativity.

To confirm with the above requirements, the third equation of (4) can only be satisfied when $\frac{dp_0}{dr} = 0$ i.e. we demand that

$$\frac{dp_0}{dr} = -\frac{1}{2}(p_0 + \rho_0)\nu' = 0 \qquad (6)$$

The equation (6) is satisfied with any of the three possibilities

$$\left.\begin{array}{ll}\text{(i)} & \nu' = 0 \\ \text{(ii)} & p_0 + \rho_0 = 0 \\ \text{(iii)} & p_0 + \rho_0 = 0, \nu' = 0\end{array}\right\} \qquad (7)$$

These three solutions lead respectively to the Einstein, the De-Sitter and the special relativity line elements of the universe and exhaust in themselves all the possibilities of a static, isotropic and homogenous universe.

5.1.1. The Einstein Universe :

Einstein line element arise from the possibility

$$\nu' = 0$$

The leads on integration

$$\nu = \text{Constant} = C_1 \text{ (say)}$$

Since for small values of r in the line element small reduce to special theory of relativity from flat space time.

i.e $\lambda = 0,\ \nu = 0$ at $r = 0$

$\therefore C_1 = 0 \Rightarrow \nu = 0$

Substituting this value of ν in the expression

$$8\pi p_0 = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda$$

We have

$$8\pi p_0 = e^{-\lambda}\frac{1}{r^2} - \frac{1}{r^2} + \Lambda$$

$$\Rightarrow e^{-\lambda} = 1 - (\Lambda - 8\pi p_0) r^2$$

$$= 1 - \frac{r^2}{R_0^{\ 2}} \qquad (2)$$

where $\Lambda - 8\pi p_0 = \dfrac{1}{R_0^{\ 2}}$ (3)

and the resulting line element is $ds^2 = -\dfrac{dr^2}{1 - \dfrac{r^2}{R_0^{\ 2}}} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + dt^2$ (4)

This line element is called Einstein line element for static isotopic and homogenous universe.

(a) Geometry of the Einstein Universe:

The line element for Einstein universe is

$$ds^2 = -\left(1-\frac{r^2}{R_0^{\,2}}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2 \qquad (1)$$

In order to understand the geometry of the space time characterized by Einstein line element, it is convenient to rewrite Einstein's line element by the transformation of co-ordinates.

(i) Consider the transformation.

$$r = \frac{\rho}{1+\frac{\rho^2}{4R_0^{\,2}}}$$

$$\Rightarrow r\left(1+\frac{\rho^2}{4R_0^{\,2}}\right) = \rho \qquad (2)$$

$$\Rightarrow dr\left(1+\frac{\rho^2}{4R_0^{\,2}}\right) + \frac{1}{2}\frac{\rho r}{R_0^{\,2}}\; d\rho = d\rho$$

$$\Rightarrow dr = \frac{\left(1-\frac{\rho r}{2R_0^{\,2}}\right)d\rho}{\left(1+\frac{\rho^2}{4R_0^{\,2}}\right)}$$

$$\Rightarrow dr^2 = \frac{\left(1-\frac{1}{2}\frac{\rho r}{R_0^2}\right)^2}{\left(1+\frac{\rho^2}{4R_0^2}\right)^2} d\rho^2$$

$$\left(1-\frac{\rho r}{2R_0^2}\right)^2 = 1+\frac{\rho^2 r^2}{4R_0^4} - \frac{\rho r}{R_0^2} = 1+\frac{r^2}{R_0^2}\,\frac{\rho^2}{4R_0^2} - \frac{\rho r}{R_0^2}$$

$$= 1+\frac{r^2}{R_0^2}\left(\frac{\rho}{r}-1\right) - \frac{\rho r}{R_0^2} \qquad \text{[using (2)]}$$

$$= 1-\frac{r^2}{R_0^2}$$

$$\therefore dr^2 = \frac{1-\frac{r^2}{R_0^2}}{\left(1+\frac{\rho^2}{4R_0^2}\right)^2} d\rho^2$$

$$\therefore \frac{dr^2}{1-\frac{r^2}{R_0^2}} = \frac{d\rho^2}{\left(1+\frac{r^2}{4R_0^2}\right)^2} \quad (3)$$

So the Einstein line element (1) transformed into

$$ds^2 = -\left(1+\frac{\rho^2}{4R_0^2}\right)^{-2} \left\{d\rho^2 - \rho^2 d\theta^2 - \rho^2 \sin^2\theta d\phi^2\right\} + dt^2 \quad (4)$$

(ii) Further substitutions

$x = \rho \sin\theta \cos\varphi, \quad y = \rho \sin\theta \sin\phi, \quad z = \rho\cos\theta$

The equation (4) leads to Einstein line element in the form.

$$ds^2 = -\left(1+\frac{\rho^2}{4R_0^2}\right)^{-2} \left(dx^2 + dy^2 + dz^2\right) + dt^2 \quad (5)$$

(iii) Again by putting

$r = R\sin\chi$ in (1) Einstein line element takes the form

$$ds^2 = -R_0^2\left(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2\right) + dt^2 \quad (6)$$

(iv) Finally considering the transformation

$$z_1 = R_0\left(1-\frac{r^2}{R_0^2}\right)^{\frac{1}{2}}, \quad z_2 = r\sin\theta\cos\phi$$

$$z_3 = r\sin\theta\sin\phi, \qquad z_4 = r\cos\theta$$

So that $z_1^2 + z_2^2 + z_3^2 + z_4^2 = R_0^2$

Thus the Einstein line element (1) takes the form

$$ds^2 = -\left(dz_1^2 + dz_2^2 + dz_3^2\right) + dt^2 \quad (7)$$

Einstein line element in this form suggests that the spatial geometry of the Einstein universe can be regarded as the immersion at a whole three dimensional spherical surface in a four dimensional Euclidean space viz.

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = R_0^2$$

This form also represents the isotropic and homogenous character of Einstein universe:

(b) Density and pressure of matter in Einstain universe :

For the general line element

$$ds^2 = -e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + e^{\nu}dt^2$$

We have,

$$8\pi\, p_0 = e^{-\lambda}\left(\frac{\nu^{1}}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \qquad (2)$$

$$8\pi\, \rho_0 = e^{-\lambda}\left(\frac{\lambda^{1}}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \qquad (3)$$

For Einstein universe

$$\nu' = 0 = \nu \text{ and } e^{-\lambda} = 1 - \frac{r^2}{R_0^2} \text{ where } \frac{1}{R_0^2} = \Lambda - 8\pi p_0$$

So that equations (2) and (3) yield

$$8\pi\, p_0 = -\frac{1}{R_0^2} + \Lambda$$

$$8\pi\, \rho_0 = \frac{3}{R_0^2} - \Lambda$$

The two unknowns Λ and R_0 can be expressed in terms of ρ_0 and p_0

$$\Lambda = 4\pi\left(3p_0 + \rho_0\right)$$

$$\frac{1}{R_0^2} = 8\pi\left(p_0 + \rho_0\right)$$

Now on physical grounds, both the proper density and proper pressure ρ_0 and p_0 are positive quantities, consequently p_0 and Λ would both be necessarily positive quantities also. If we regard Λ and R_0^2 as arbitrary parameters a given distribution of matter fluid having known ρ_0 and p_0 determine these parameters.

(c) Behaviour of particles and light rays in Einstein universe:

Einstein line element is

$$ds^2 = -\left(1 - \frac{r^2}{R_0^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2 \qquad (1)$$

The motion of a test particle in the gravitational field corresponding to line element (1) would be described by geodesic equations.

$$\frac{d^2x^{\alpha}}{ds^2} + \Gamma^{\alpha}{}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0 \qquad (2)$$

For the sake of simplicity let the test particle be initially at rest, so that the components of the spatial velocity of test particle are zero i.e..

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0$$

The equation (2) becomes

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{44}\left(\frac{dt}{ds}\right)^2 = 0 \qquad (3)$$

But $\Gamma^\alpha_{44} = \frac{1}{2}\frac{\partial g_{44}}{\partial x^\alpha} = 0$, since $g_{44} = 1$

$$\therefore (3) \Rightarrow \frac{d^2r}{ds^2} = \frac{d^2\theta}{ds^2} = \frac{d^2\phi}{ds^2} = 0$$

i.e. the particle has zero acceleration. Hence in Einstein universe a rest particle would remain permanently at rest. We may also interpret that matter in Einstein universe is without motion.

(d) Doppler's effect in Einstein universe:

Consider an observer situated at $r = 0$ and a source of light, say a star at $r = r_1$ both being permanently at rest with respect to spatial co-ordinates in accordance with zero acceleration for stationary particles.

For a light ray emitted from the star travelling along the radial direction we have

$$ds = 0, \qquad d\theta = d\phi = 0$$

So that from Einstein line element

$$ds^2 = -\left(1 - \frac{r^2}{R_0^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + dt^2 \qquad (1)$$

where $\frac{1}{R_0^2} = \Lambda - 8\pi p_0$

The radial velocity of light from or towards the origin is given by

$$\frac{dr}{dt} = \pm\left(1 - \frac{r^2}{R_0^2}\right)^{\frac{1}{2}} \qquad (2)$$

Let a light pulse leave the star at time t_1 it would reach the observer at time t_2

$$\therefore \int_{t_1}^{t_2} dt = -\int_{r_1}^{0} \frac{dr}{\left(1 - \frac{r^2}{R_0^2}\right)^{\frac{1}{2}}} = \int_{0}^{r_1} \frac{dr}{\sqrt{1 - \frac{r^2}{R_0^2}}}$$

$$\Rightarrow t_2 - t_1 = R_0 \sin^{-1}\left(\frac{r_1}{R_0}\right)$$

$$\Rightarrow t_2 = t_1 + R_0 \sin^{-1}\left(\frac{r_1}{R_0}\right) \tag{3}$$

Since in Einstein Universe the matter has no motion, the star remains at rest i.e. r_1 is constant.

So differentiating (3) we have

$$\delta t_2 = \delta t_1 \tag{4}$$

i.e. the time interval δt_2 between the reception of two successive wave creat at the origin is equal to the interval δt_1 between their emission.

According to Einstein line element (1) the co-ordinate time *t* is identical to the proper time as measured by the local observer at rest with respect to the spatial co-ordinates ($dt = ds$) this equality also implies the equality of the proper periods of the emitted and received light as measured by the observers at rest with respect to the original source and at rest at the origin. Consequently the wave length of light at emission and at reception would be the same. Thus there is no shift of spectral line i.e. no Doppler effect is observed in Einstein Universe.

It contradicts the actual universe where, according to Hubble and Humason a definite red shift is observed in light from the nebulac, which (red shift) increases at least very closely in linear propagation with distance.

5.1.2. The De-sitter universe :

The pressure and density p_0 and ρ_0 are determined by field equations (G = C = 1)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \tag{1}$$

w. r to the line element

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \tag{2}$$

are given by

$$8\pi p_0 = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \tag{3}$$

$$8\pi \rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \tag{4}$$

For static, isotropic and homogenous universe we have the following three possibilities.

(i) $\nu' = 0$ (Einstein Universe)

(ii) $p_0 + \rho_0 = 0$ (De-sitter Universe)

(iii) $p_0 + \rho_0 = 0, \nu' = 0$ (Special relativity line element)

The De-sitter line element arises from

$$p_0 + \rho_0 = 0 \tag{5}$$

According to (3) and (4) we get

$$8\pi(p_0 + \rho_0) = e^{-\lambda} \frac{\lambda' + \nu'}{r}$$

using (5) we have

$$\lambda' + \nu' = 0 \tag{6}$$

Integrating

$$\lambda + \nu = \text{costant}(c)$$

Applying the boundary condition that at $r = 0$, $\lambda = 0 = \nu \Rightarrow c = 0$

$$\therefore \lambda = -\nu \tag{7}$$

The equation (4) may be written as

$$e^{-\lambda}(\lambda' r - 1) = (8\pi\rho_0 + \Lambda) r^2 - 1$$

$$\Rightarrow \frac{d}{dr}(re^{-\lambda}) = 1 - (8\pi\rho_0 + \Lambda) r^2$$

Intergrating we get

$$re^{-\lambda} = r - \frac{8\pi\rho_0 + \Lambda}{3} r^3 + C_1$$

Applying the boundary conditions at $r = 0, \lambda = 0 = \nu$ we get C_1.

$$\therefore re^{-\lambda} = r - \frac{8\pi\rho_0 + \Lambda}{3} r^3$$

or $$e^{-\lambda} = 1 - \frac{8\pi\rho_0 + \Lambda}{3} r^2$$

Substituting $\frac{1}{R_0^2} = \frac{8\pi\rho_0 + \Lambda}{3}$ we get

$$e^{-\lambda} = 1 - \frac{r^2}{R_0^2}$$

So the line element (1) becomes

$$ds^2 = -\left(1 - \frac{r^2}{R_0^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \left(1 - \frac{r^2}{R_0^2}\right) dt^2$$

This line element was discovered by de-sitter and is called de-sitter line element for static, isotropic and homogeneous universe.

(a) Geometry of de-sitter universe:

In order to under stand the geometry of space time characterized by de-sitter line element, it is convenient to re-write de-sitter line element by the transformation of co-ordinates.

The de-sitter line element is

$$ds^2 = -\left(1-\frac{r^2}{R_0^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + \left(1-\frac{r^2}{R_0^2}\right) dt^2 \quad (1)$$

(i) Considering the transformation $r = R_0 \sin\chi$

The de-sitter line element transform to

$$ds^2 = -R_0^2\left(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin\theta d\varphi^2\right) + \cos^2\chi dt^2 \quad (2)$$

(ii) For further simplification of de-sitter line-element, let us substitute $\alpha = r\sin\theta\cos\phi$, $\beta = r\sin\theta\sin\phi$, $\gamma = r\cos\theta$

$$\delta + \in = R_0 e^{\frac{t}{R_0}}\sqrt{1-\frac{r^2}{R_0^2}}, \qquad \delta - \in = R_0 e^{\frac{t}{R_0}}\sqrt{1-\frac{r^2}{R_0^2}} \quad (3)$$

So that $d\alpha^2 + d\beta^2 + d\gamma^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$

$$d\left(\delta^2 - \in^2\right) = d(\delta + \in)\, d(\delta - \in)$$

$$= -\left(1-\frac{r^2}{R_0^2}\right) dt^2 + \frac{r^2}{R_0^2\left(1-\frac{r^2}{R_0^2}\right)} dr^2 \quad (4)$$

Hence de-sitter line element (1) is reduced to

$$ds^2 = -d\alpha^2 - d\beta^2 - d\gamma^2 - d\delta^2 + d\in^2 \quad (5)$$

(iii) Further substituting in (5)

$z_1 = i\alpha,\quad z_2 = i\beta,\quad z_3 = i\gamma,\quad z_4 = i\delta,\quad z_5 = \in$

we get $\quad ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 + dz_5^2$

with $\quad z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = (iR_0)^2 \quad (6)$

The above equation determines that four dimensional surface in the five dimensional manifold which correspond to the space time and we may regard the geometry of de-sitter universe as that holding on the surface of a sphere embedded in four dimensional Euclidean space.

(b) Properties of the de-sitter universe, Absence of matter and radiation:

The de-sitter line element results from the requirement.

$$p_0 + \rho_0 = 0 \quad (1)$$

Now on physical grounds the proper density ρ_0 can either zero or greater than zero i.e. $\rho_0 \geq 0$. Hence equation (1) will be satisfied only if

$$\rho_0 = 0, p_0 = 0 \tag{2}$$

This amount to say that the de-sitter universe is completely empty. It does not contain any matter or radiation in any form whatever.

With this we can write the de-sitter line element as

$$ds^2 = -\frac{dr^2}{1-\frac{r^2}{R_0^2}} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + \left(1-\frac{r^2}{R_0^2}\right)dt^2 \tag{3}$$

where $\frac{1}{R_0^2} = \frac{\Lambda}{3}$

But the above equation does not fix up the sign of the cosmological constant Λ. Depending on whether the cosmological constant Λ is positive, zero or negative, we have for the line element.

(i) $ds^2 = -\frac{dr^2}{1-\frac{\Lambda}{3}r^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + \left(1-\frac{\Lambda}{3}r^2\right)dt^2, \quad \Lambda > 0$

(ii) $ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + dt^2; \quad \Lambda = 0$

(iii) $ds^2 = -\frac{dr^2}{1+\frac{\Lambda}{3}r^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + \left(1+\frac{\Lambda}{2}r^2\right)dt^2, \quad \Lambda < 0$

The models represented by the line element (i), (ii) and (iii) are called respectively the spatially closed, flat and spatially open but curved.

(c) Behaviour of test particles in a de-sitter universe :

The de-sitter line element is

$$ds^2 = -e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + e^{\nu}dt^2 \tag{1}$$

with $e^{-\lambda} = e^{\nu} = 1-\frac{r^2}{R_0^2}, \; \frac{1}{R_0^2} = \frac{8\pi\rho_0 + \Lambda}{3}$

The motion of test particle is governed by the geodesic equations.

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{2}$$

with $x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t$

The non vanishing christoffel's symbol of 2nd kind corresponding to line element (1) are

$$\left.\begin{array}{l}\Gamma^1_{11}=\frac{\lambda'}{2}, \Gamma^1_{22}=-re^{-\lambda}, \Gamma^1_{33}=-re^{-\lambda}\sin^2\theta \\ \Gamma^1_{44}=\frac{1}{2}v'e^{v-\lambda}, \Gamma^2_{12}=\Gamma^3_{13}=\frac{1}{r} \\ \Gamma^2_{33}=-\sin\theta\cos\theta, \Gamma^3_{23}=\cot\theta, \Gamma^4_{14}=\frac{v'}{2}\end{array}\right\} \tag{3}$$

For $\alpha = 1, 2, 3, 4$ geodesic equation (2) are written as

$$\frac{d^2x^1}{ds^2}+\Gamma^1_{11}\left(\frac{dx^1}{ds}\right)^2+\Gamma^1_{22}\left(\frac{dx^2}{ds}\right)^2+\Gamma^1_{33}\left(\frac{dx^3}{ds}\right)^2+\Gamma^1_{44}\left(\frac{dx^4}{ds}\right)^2=0$$

$$\frac{d^2x^2}{ds^2}+\Gamma^1_{12}\frac{dx^1}{ds}\frac{dx^2}{ds}+\Gamma^2_{21}\frac{dx^2}{ds}\frac{dx^1}{ds}+\Gamma^2_{33}\left(\frac{dx^3}{ds}\right)^2=0$$

$$\frac{d^2x^3}{ds^2}+\Gamma^3_{13}\frac{dx^1}{ds}\frac{dx^3}{ds}+\Gamma^3_{31}\frac{dx^3}{ds}\frac{dx^1}{ds}+\Gamma^3_{23}\frac{dx^2}{ds}\frac{dx^3}{ds}+\Gamma^3_{32}\frac{dx^3}{ds}\frac{dx^2}{ds}=0$$

$$\frac{d^2x^4}{ds^2}+\Gamma^4_{14}\frac{dx^1}{ds}\frac{dx^4}{ds}+\Gamma^4_{41}\frac{dx^4}{ds}\frac{dx^1}{ds}=0$$

Using (3) above equations give

$$\frac{d^2r}{ds^2}+\frac{1}{2}\lambda^1\left(\frac{dr}{ds}\right)^2-re^{-\lambda}\left(\frac{d\theta}{ds}\right)-r\sin^2\theta e^{-\lambda}\left(\frac{d\varphi}{ds}\right)^2+\frac{1}{2}v'e^{v-\lambda}\left(\frac{dt}{ds}\right)^2=0 \tag{4}$$

$$\frac{d^2\theta}{ds^2}+\frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds}-\sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)=0 \tag{5}$$

$$\frac{d^2\phi}{ds^2}+\frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds}+2\cot\frac{d\theta}{ds}\frac{d\phi}{ds}=0 \tag{6}$$

$$\frac{d^2t}{ds^2}+v'\frac{dr}{ds}\frac{dt}{ds}=0 \tag{7}$$

To integrate these equations we choose the initial motion to be in the place

$\theta=\frac{\pi}{2}$ then -

$$\sin\theta=1, \cos\theta=0, \frac{d\theta}{ds}=0 \tag{8}$$

The equation $(5)\Rightarrow\frac{d^2\theta}{ds}=0$ (9)

This equation implies that the particle will continue to move in the plane $\theta = \frac{\pi}{2}$ using (8) in (4), (6) and (7)

$$\frac{d^2r}{ds^2} + \frac{1}{2}\lambda'\left(\frac{dr}{ds}\right) - re^{-\lambda}\left(\frac{d\varphi}{ds}\right) + \frac{1}{2}\nu' e^{\nu-\lambda}\left(\frac{dt}{ds}\right) = 0 \tag{10}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0 \tag{11}$$

$$\frac{d^2t}{ds^2} + \nu'\frac{dr}{ds}\frac{dt}{ds} = 0 \tag{12}$$

$$(11) \Rightarrow \frac{1}{r^2}\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0$$

$$\Rightarrow \frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0 \tag{13}$$

$$(12) \Rightarrow \frac{1}{e^\nu}\frac{d}{ds}\left(e^\nu\frac{dt}{ds}\right) = 0$$

$$\Rightarrow \frac{d}{ds}\left(e^\nu\frac{dt}{ds}\right) = 0 \tag{14}$$

Integrating (13) and (14) we have

$$r^2\frac{d\phi}{ds} = h \Rightarrow \frac{d\phi}{ds} = \frac{h}{r^2} \tag{15}$$

$$e^\nu\frac{dt}{ds} = K \Rightarrow \frac{dt}{ds} = K\left(1 - \frac{r^2}{R_0^2}\right)^{-1} \tag{16}$$

where h and K are constants of integration.

The constant h is a measure of angular momentum of the motion. Further instead of working with equation (10) due to trouble some integration, we use the line element (1) which by use of (8) yields.

$$e^\lambda\left(\frac{dr}{ds}\right) + r^2\left(\frac{d\phi}{ds}\right)^2 - e^\nu\left(\frac{dt}{ds}\right)^2 + 1 = 0$$

Using (15) and (16) we have,

$$\left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{r^2}{R_0^2}\right) - K^2 + \left(1 - \frac{r^2}{R_0^2}\right) = 0$$

or $$\left(\frac{dr}{ds}\right)^2 = K^2 - 1 + \frac{r^2}{R_0^2} - \frac{h^2}{r^2} + \frac{h^2}{R_0^2}$$

or $\left(\frac{dr}{ds}\right)=\pm\sqrt{K^2-1+\frac{r^2}{R_0^2}+\frac{h^2}{R_0^2}-\frac{h^2}{r^2}}$

Note that the parameter h and K can be both positive and negative depending on the direction of motion, K is positive for $r < R_0$, since t is related to proper time ds.

$$\frac{dr}{ds}=\frac{dr}{d\phi}\frac{d\phi}{ds}=\sqrt{K^2-1+\frac{r^2}{R_0^2}+\frac{h^2}{R_0^2}-\frac{h^2}{r^2}}$$

$$\Rightarrow \frac{dr}{d\phi}.\frac{h}{r^2}=\left\{K^2-1+\frac{r^2}{R_0^2}+\frac{h^2}{R_0^2}-\frac{h^2}{r^2}\right\}^{\frac{1}{2}}$$

Putting, $r=\frac{1}{u}$

$$-h\frac{du}{d\varphi}=\left\{K^2-1+\frac{r^2}{R_0^2}+\frac{h^2}{R_0^2}-\frac{h^2}{r^2}\right\}^{\frac{1}{2}}$$

$$\Rightarrow h^2\left(\frac{du}{d\phi}\right)^2=K^2-1+\frac{1}{R_0^2u^2}+\frac{h^2}{R_0^2}-h^2u^2$$

Differentiating w.r.t. ϕ and simplifying

We have $\frac{d^2u}{d\phi^2}+u=-\frac{1}{h^2R_0^2u^3}=-\frac{1}{3}\frac{\Lambda}{h^2u^2}$

This is the equation of the orbit of a particle in the de-sitter universe and corresponds in Newtonian mechanics to the motion of particle with a central impulsive force proportional to the distance r, when $\Lambda > 0$

(d) Velocity and Acceleration of the particle in de-sitter universe :

We have $\frac{dr}{dt}=\frac{dr}{ds}\frac{ds}{dt}=\frac{1}{K}\left(1-\frac{r^2}{R_0^2}\right)\left[K^2-1+\frac{r^2}{R_0^2}-\frac{h^2}{r^2}+\frac{h^2}{R_0^2}\right]^{\frac{1}{2}}$ (17)

and $\frac{d\phi}{dr}=\frac{d\phi}{ds}\frac{ds}{dr}=\frac{h}{Kr^2}\left(1-\frac{r^2}{R_0^2}\right)$ (18)

The radial velocity $\frac{dr}{dt}$ is zero, when either

$$r=R_0 \text{ or } K^2-1+\frac{r^2}{R_0^2}-\frac{h^2}{r^2}+\frac{h^2}{R_0^2}=0 \quad (19)$$

Equation (19) determines the value of r at perihelion.

The velocity component $\frac{d\phi}{dt}$ is zero when $r = R_0$

$\therefore \frac{dr}{dt} = \frac{d\phi}{dt} = 0$ at $r = R_0$ i.e. all motion ceases at $r = R_0$. This value of r is called the apparent horizon of the universe.

Differentiating (17) and (18) we get

$$\frac{d^2r}{dt^2} = -\frac{\frac{2r}{R_0}}{\left(1-\frac{r^2}{R_0^2}\right)}\left(\frac{dr}{dt}\right) + \frac{1}{K^2}\left(1-\frac{r^2}{R_0^2}\right)^2\left(\frac{r}{R_0^2}+\frac{h^2}{r^3}\right) \tag{20}$$

$$\frac{d^2\phi}{dt^2} = -\frac{2h}{Kr^3}\frac{dr}{dt} \tag{21}$$

From (20) it is obvious that for a particle with radial velocity zero, the radial acceleration $\frac{d^2r}{dt^2}$ at a point r where $0 \le r \le R_0$ is necessarily positive. Hence a free particle after reaching the perihelion starts to turn away from the origin and would never return. Also for a particle at rest at the origin with $r = 0,\ h = 0$, the acceleration would be zero and if would remain at rest for ever.

(e) Doppler effect in de-sitter universe:

Consider an observer situated in the origin $r = 0$ and the source light, say a star at $r = r$ in de-sitter universe.

For a light emitted from the star travelling along the radial direction,

we have $ds = 0, \quad d\theta = d\phi = 0$

So that $0 = -\left(1-\frac{r^2}{R_0^2}\right)^{-1} dr^2 + \left(1-\frac{r^2}{R_0^2}\right) dt^2$

$$\Rightarrow \frac{dr}{dt} = \pm\left(1-\frac{r^2}{R_0^2}\right) \tag{1}$$

Let the star emit the light pulse at time t it would reach the observer at time t' given by

$$\int_t^{t'} dt = -\int_r^0 \frac{dr}{\left(1-\frac{r^2}{R_0^2}\right)} = \int_0^r \frac{dr}{1-\frac{r^2}{R_0^2}}$$

$$\Rightarrow t'-t = \int \frac{dr}{1-\frac{r^2}{R_0^2}} \tag{2}$$

Now let δt be the time interval of the emission of two successive wave crests at $r = r$ and $\delta t'$ be the corresponding time of their reception by an observer at rest at the origin.

Then differentiations of (2) yields $\delta t' = \delta t + \frac{1}{1-\frac{r^2}{R_0^2}} \frac{dr}{dt} dt$

which $\frac{dr}{dt}$ is the radial velocity of the source at the time emission.

Further the proper time δs for an observer on the moving particle and the corresponding co-ordinate time δt assuming the motion to be in the plane $\theta = \frac{\pi}{2}$ are related by

$$\delta s = \frac{1}{K}\left\{1-\frac{r^2}{R_0^2}\right\}\delta t \tag{3}$$

while the proper time interval between successive crests as measure at the origin in $\delta t'$. Thus there is a change in the frequency at the time of emission and reception. If ν and ν' are frequencies at the time of emission and at reception respectively, thus we have.

$$\nu' \delta t' = \nu \delta t$$

i.e. $\nu' = \nu \frac{ds}{dt'} = \nu \frac{ds}{dt}\frac{dt}{dt'}$

$$\left. = \frac{1}{K}\left\{1-\frac{r^2}{R_0^2}\right\} \times \frac{1}{1+\frac{1}{1-\frac{r^2}{R_0^2}}\frac{dr}{dt}} \nu \right\} \tag{4}$$

The Doppler's shift in spectral lines is given by

$$\frac{\delta \nu}{\nu} = \frac{\nu'-\nu}{\nu} = \frac{1}{K}\left\{1-\frac{r^2}{R_0^2}\right\} - \frac{1}{1-\frac{r^2}{R_0^2}}\frac{dr}{dt} \tag{5}$$

If λ and λ' are the wave length at emission and reception respectively we have from (5) using $c = \nu\lambda$.

$$\frac{\lambda'}{\lambda} = \frac{K}{1-\frac{r^2}{R_0^2}} + \frac{K}{\left\{1-\frac{r^2}{R_0^2}\right\}^2}\frac{dr}{dt} \qquad (6)$$

Since K is positive and $1-\frac{r^2}{R_0^2} > 0$,

(i) When $\frac{dr}{dt} > 0$ then $\frac{\lambda'}{\lambda} > 0$ i.e $\lambda' > 0$. This means there is red shift.

(ii) When $\frac{dr}{dt} < 0$, then $\frac{\lambda'}{\lambda}$ may be greater or less than zero depending upon the magnitude of velocity of the distance source.

Thus in this case, there may be red or violet shift depending upon the magnitude of the velocity of the source.

The violet shift is only possible when the magnitude of negative $\frac{dr}{dt}$ is sufficiently large to make R.H.S. of (6) to be negative.

(iii) At the perihelion $\frac{dr}{dt} = 0$ we get

$$\frac{\lambda'}{\lambda} = \frac{K}{\left\{1-\frac{r^2}{R_0^2}\right\}} = \frac{\sqrt{1-\frac{r^2}{R_0^2}+\frac{h^2}{r^2}-\frac{h^2}{R_0^2}}}{\left\{1-\frac{r^2}{R_0^2}\right\}} \qquad (7)$$

which is positive. Thus there is a red shift.

Thus we see that in the de-sitter universe there may be both red or violet shifts, but the possibility of red shift is more prominent.

5.2. Non Static Cosmological Models :

The models represented by Einstein and de-sitter universe are static solutions of the Einstein modified filed equations.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Both these solutions requires the cosmological constant Λ to be positive real constant greater than zero and it determines the curvature of the space. In the limit $\Lambda = 0$ we obtain a third model in classical infinite Euclidean space. This model is empty and the space time is that of special relativity. These three models exhaust in themselves all those solutions of the above field equations which admit static, isotropic and homogenous distribution of matter in the universe.

To go beyond the Einstein and de-sitter models, one should discard some of the restrictions imposed earlier which led to the above solutions. There are however reasons to believe is at the universe is isotropic and homogeneous in nature from a large scale point of view. One should therefore retain the hypotheses of isotropy and homogeneity of the universe and relax the requirement of static nature in order to obtain more general solutions.

Further progress in studying these aspects is associated with three names:

Friedmann (1922), Lemaitre (1972) and Robertson (1929), Each of them tried to investigate the same problems viz. "what is the most general quadratic line element in the four dimensional manifold of space time which would describe a non static but isotropic and homogenous universe?" Each of them succeeded in getting the answer to the above problem.

5.2.1. Mach's Principle :

According to Einstein, March's Principle (1883) denotes "the general idea that the geometry of space time is determined by the distribution of matter and energy so that some kind of field equations connecting the components of the material tensor $g_{\mu\nu}$ with those of the energy momentum tensor, $T_{\mu\nu}$ are in any case implied" as explicitly shown by Einstein field equations.

It should be mentioned that Mach's Principle has been defined by different authors in different forms. However, Mach's principle, as understand by Einstein inspired him to formulated his new theory of gravitation.

5.2.2. Cosmological Principle :

The Cosmological Principle is stated in the following form :-

At each epoch (t), the universe is homogeneous and isotropic.

This means that at each epoch, the universe has the same property at every point in the space like hypersurface and in every direction about any point.

Other wise the hypersurface has no privileged points and has no privileged directions about any point.

There astronomical support for this principle. In the first place, the spatial homogeneity is supported by the fairly uniform distribution of galaxies on the one hand, and the linearity of the Hubble law on the other. In the second place in 1965 Penzias and Wilson discovered the isotropic character of the cosmic background radiation. This is indeed the greatest evidenced for the isotropy of the universe.

5.2.3. Weyl Postulate :

This postulate, given H. Weyl, deals with how the universe evolves with time (t) where as the cosmological principle indicates the picture or state of the universe at a certain epoch (t). The evolution, according to his postulate is conceived as follows. The galaxies of the universe are considered as a particle in a perfect fluid. The three dimensional space containing the particle is regarded as a hypersurface orthogonal to time. Figure 5.2 shows how the hypersurface (s) is evolving with time (t) . It is envisaged that the particles in a hypersurface travel along time-oriented geodesics, each geodesic being given by

$$x_1, x_2, x_3 = \text{constant}$$

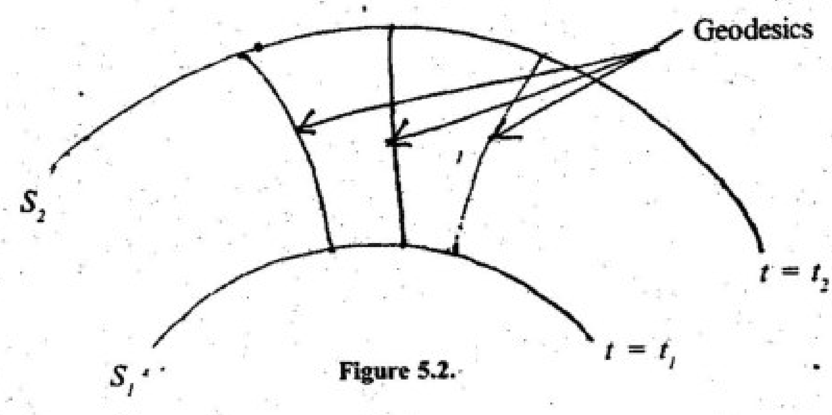

Figure 5.2.

where x_1, x_2, x_3 define the space co-ordinates of a particle. These co-ordinates are carried by a particle all along the geodesic and are called the commoving co-ordinates. The time t being the same for all particles (galaxies) in a hyper surface (Fig. 5.2.) is called the cosmic time. The geodesics start from a point in the past when $t = 0$ and proceed as lives defined by the commoving co-ordinates of the particles. The geodesics as defined here do not intersect.

The weyl postulate can now be stated incorporating the above ideas.

The particles (galaxies) in the evolving hyper surface lie in space time on a congruence (bundle) of time oriented geodesics diverging from a point in the past and orthogonal to the evolving hyper surface.

5.3. Derivation of the Robertson walker line element :

Robertson Walker derive the line element of the non-static but isotropic and homogenous universe under the following hypothesis:

(i) There exists a cosmic time which is orthogonal to the spatial geometry.

$$ds^2 = dt^2 + g_{\mu\nu} dx^\mu \, dx^\nu \qquad (1)$$

(ii) The three dimensional spatial surfaces belonging to t = constant are locally isotropic and homogeneous.

For convenience let us express this line element in spherical polar co-ordinates. The non static, spherically symmetric line element in moving co-ordinate system is given by

$$ds^2 = dt^2 - e^{\mu(r,t)} \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right] \qquad (2)$$

where μ is a function of r and t. The assumption of isotropy and homogeneity restrict the form of $\mu(r,t)$ as

$$\mu(r,t) = f(r) + g(t)$$

Therefore the line element (2) may be expressed as

$$ds^2 = -e^{f(r)+g(t)} \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right] + dt^2$$

Writing $\dot{\mu} = \frac{\partial u}{\partial t}$ and $\mu' = \frac{\partial u}{\partial r}$ the serving christoffel's symbol's are

$$\Gamma^4_{11} = \frac{1}{2} e^\mu \dot{\mu} \, \Gamma^2_{42} = \frac{\dot{\mu}}{2}, \Gamma^4_{22} = \frac{1}{2} r^2 e^\mu \dot{\mu}$$

$$\Gamma^2_{12} = \frac{1}{r} + \frac{1}{2}\mu', \Gamma^4_{33} = \frac{1}{2} r^2 \sin^2\theta e^\mu \dot{\mu}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta, \; \Gamma^1_{41} = \frac{1}{2}\dot{\mu} = \Gamma^3_{43} = \Gamma^3_{34}$$

$$\Gamma^1_{11}=\frac{1}{2}\mu', \ \Gamma^3_{13}=\frac{1}{r}+\frac{1}{2}\mu' \ \Gamma^3_{23}=\cot\theta$$

$$\Gamma^1_{22}=-\left(r+\frac{1}{2}r^2\mu'\right), \ \Gamma^1_{33}=-\left(r+\frac{1}{2}r^2\mu'\right)\sin^2\theta$$

A detailed calculation of the contracted Riemann Christoffel tensor $R_{\mu\nu}$ now leads to

$$\left.\begin{aligned}
R_{11}&=f''+\frac{1}{r}f'-e^{\mu}\left(\frac{1}{2}\ddot{g}+\frac{3}{4}\dot{g}^2\right)\\
R_{22}&=r^2\left[\frac{1}{2}f''+\frac{1}{4}f'^2+\frac{3}{2r}f'-e^{\mu}\left(\frac{1}{2}\ddot{g}+\frac{3}{4}\dot{g}^2\right)\right]\\
R_{33}&=\sin^2\theta\, R_{22}\\
R_{44}&=\frac{3}{2}\ddot{g}+\frac{3}{4}\dot{g}^2
\end{aligned}\right\} \qquad (4)$$

and $R_{\mu\nu}=0$ for $\mu\neq\nu$

Since $R^{\mu}_{\nu}=g^{\mu\alpha}R_{\nu\alpha}$

$$\therefore R^1{}_1=\frac{1}{2}\ddot{g}+\frac{3}{4}\dot{g}^2-e^{-\mu}\left\{f''+\frac{f'}{r}\right\}$$

$$R^2{}_2=\frac{1}{2}\ddot{g}+\frac{3}{4}\dot{g}^2-e^{-\mu}\left[\frac{1}{2}f''+\frac{1}{4}f'^2+\frac{3}{2r}f'\right]$$

$$R^3_3=R^2{}_2$$

$$R^4{}_4=\frac{3}{2}\ddot{g}+\frac{3}{4}\dot{g}^2$$

and the scalar curvature

$$R=R^{\mu}{}_{\nu}=3\left(\ddot{g}+\dot{g}^2\right)-2e^{-\mu}\left(f''+\frac{1}{4}f'^2+\frac{2}{r}f'\right)$$

The field equations are

$$R^{\mu}{}_{\nu}-\frac{1}{2}g^{\mu}{}_{\nu}R+\Lambda g^{\mu}{}_{\nu}=-8\pi\, T^{\mu}{}_{\nu}$$

where $G=c=1$

Now lead to

$$\left.\begin{aligned}
8\pi T_1^1 &= -e^{-\mu}\left(\frac{1}{4}f'' + \frac{1}{r}f'\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \\
8\pi T_2^2 = 8\pi T_3^3 &= -e^{-\mu}\left(\frac{1}{2}f'' + \frac{1}{2r}f'\right) + \ddot{g} + \frac{3}{4}\dot{g}^2 - \Lambda \\
8\pi T_4^4 &= -e^{-\mu}\left(f'' + \frac{1}{4}f'^2 + \frac{2}{r}f'\right) + \frac{3}{4}\dot{g}^2 - \Lambda
\end{aligned}\right\} \qquad (5)$$

$$8\pi T_\nu^\mu = 0 \quad \text{for} \quad \mu \neq \nu$$

The spatial isotropy of the 3 space requires that the longitudinal and the transverse stresses be equal i.e.

$$T_1^1 = T_2^2 = T_3^3$$

using this result in (5) we have

$$f'' - \frac{1}{2}f'^2 - \frac{1}{r}f' = 0$$

A first integral of this equation is

$$\frac{df}{dr} = K_1 r e^{\frac{1}{2}f}, \quad K_1 = \text{Constant}$$

A second integration gives

$$e^{f(r)} = \frac{\frac{1}{K_2^2}}{\left(1 - \frac{K_1}{K_2}\frac{r^2}{4}\right)^2}, \quad K_2 = \text{constant}$$

Finally we put

$$\frac{-K_1}{K_2} = \frac{1}{R_0^2}, \text{ where } R_0 \text{ is a constant, may be } +ve, -ve \text{ or infinite}$$

$$\therefore e^{f(r)} = \frac{\frac{1}{K_2^2}}{\left(1 + \frac{r^2}{4R_0^2}\right)^2} \cong \frac{1}{\left(1 + \frac{Kr^2}{4R_0^2}\right)^2}$$

Thus the line element finally we get.

$$ds^2 = -\frac{e^{g(t)}}{\left(1 + \frac{Kr^2}{4R_0^2}\right)^2}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right) + dt^2$$

where $K = 1, 0, -1$ corresponding to whether the constant R_0^2 is positive, infinite or negative.

This line element is known as Robertson Walker (R – W) line element.

Again putting

$$r_1 = \frac{r}{1 + \frac{r^2}{4R_0^2}} \quad \text{for} \quad K = +1$$

The R – W line element becomes

$$ds^2 = -e^{g(t)} \left[\frac{dr_1^2}{1 - \frac{r_1^2}{R_0^2}} + r_1^2 d\theta^2 + r_1^2 \sin^2 \theta d\varphi^2 \right] + dt^2$$

$$\Rightarrow ds^2 = -s^2(t) \left[\frac{dr^2}{1 - \frac{r^2}{R_0^2}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right] + dt^2 \quad (6)$$

Derivation of Hubble's Law :

If the co-ordinate distance between two galaxies, P and Q is finite and is equal to ℓ then the physical distance between their from (6) is

$$D = s(t)\,\ell \quad (7)$$

Relative velocity of one galaxy w. r. to the other is $V = \frac{dD}{dt} = \dot{s}\,\ell = \frac{\dot{s}}{s}(s\ell) = HD$

where $H = \frac{\dot{s}}{s} =$ Hubble constant

$\therefore V \propto D$ (Hubble's law).

5.4. Red Shift in R-W line element:

Let us show that R-W line element predicts a shift in the wave length of radiation emitted by a distant source such as a nebula or a radio galaxy. The R-W line element is

$$ds^2 = dt^2 - \frac{R^2(t)}{\left\{1 + \frac{Kr^2}{4}\right\}^2} \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right) \quad (1)$$

where $e^{g(t)} = R^2(t)$ and $R = R_0 e^{-\frac{1}{2}g(t)}$

Recalling that a world line of a light ray is along the geodesic $ds = 0$ and from symmetry considerations it is evident that the light ray will propagate radially, i.e. along a line $\theta =$ constant, $\phi =$ constant. Thus setting $d\theta = 0$, and $d\varphi = 0$ in (1) we get

$$\left(\frac{dr}{dt}\right)^2 = \frac{\left(1+\frac{Kr^2}{4}\right)^2}{R^2(t)}$$

i.e. $$\frac{dr}{dt} = \pm\frac{1+\frac{Kr^2}{4}}{R(t)} \qquad (2)$$

Here positive sign applies when light travels away from origin and minus sign applies when it travels towards the origin.

Let us now consider an observer located at the origin 'o' of our co-ordinate system and let a light source (e.g. nebula) located at any point ($\theta, \varphi =$ constant) emit two successive light pulse at times t_0 and $t_0 + \Delta t_0$. Let these puse be received by observer, at 'o' at instants t and $t + \Delta t$ respectively. Obviously $t > t_0$ and $t + \Delta t > t_0 + \Delta t_0$

Now from (2) we may write

$$\int_{t_0}^{t} \frac{dt}{R(t)} = -\int_{r}^{0} \frac{dr}{1+\frac{Kr^2}{4}}$$

and $$\int_{t_0+\Delta t_0}^{t+\Delta t} \frac{dt}{R(t)} = -\int_{r}^{0} \frac{dr}{1+\frac{Kr^2}{4}} \qquad (3)$$

Further we may write $$\int_{t_0+\Delta t_0}^{t+\Delta t} \frac{dt}{R(t)} = \left[\int_{t_0+\Delta t_0}^{t_0} + \int_{t_0}^{t+\Delta t}\right] \frac{dt}{R(t)}$$

$$= -\frac{\Delta t_0}{R(t_0)} + \int_{t_0}^{t} \frac{dt}{R(t)} + \frac{\Delta t}{R(t)} \qquad (4)$$

Using (4), equation (3) lead to

$$\frac{\Delta t}{R(t)} = \frac{\Delta t_0}{R(t_0)} \qquad (5)$$

Thereby indicating that two intervals Δt_0 of emission and Δt of reception are not equal unless $R(t)$ is constant.

Further the proper time of emission $\Delta\tau_0$ (say) is equal to the co-ordinate time Δt_0, i.e. $\Delta\tau_0 = \Delta t_0$. Similarly the proper time $\Delta t'$ of reception is equal to the co-ordinate time $\Delta\tau$. Let n be the number of waves emitted in a proper time interval $\Delta\tau$ having a proper frequency ν_0, These n waves are received with different frequency ν; thus we have

$$n = \nu_0 \Delta t_0 = \nu \Delta t \qquad (6)$$

i.e. $$\frac{\nu_0}{\nu} = \frac{\Delta t}{\Delta t_0} \qquad (7)$$

As $\nu = \frac{c}{\lambda}$ and $\nu_0 = \frac{c}{\lambda_0}$, we may write (7) as

$$\frac{\lambda}{\lambda_0} = \frac{\Delta t}{\Delta t_0} = \frac{R(t)}{R(t_0)} \quad \text{using (5)}$$

Introducing a new parameter z, such that $z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{d\lambda}{\lambda}$.

we get $$1 + z = \frac{\Delta t}{\Delta t_0} = \frac{R(t)}{R(t_0)}$$

As $\Delta t > \Delta t_0$; $R(t) > R(t_0)$ this implies z is a positive number, i.e. light emitted from a distant nebula shows shift towards the red end of the spectrum, which is also an experimental observation. The shift calculated by R-W line element agrees with experimental value.

Conversely we may say since experimental observation show a shift towards the red end of the spectrum, (i.e. z is positive); this implies $R(t) > R(t_0)$ i.e. radins $R(t)$ of the universe is increasing. In other words non static, isotropic and homogenous model of universe reveals that the universe is expanding

Exercise

1. Obtain the line elements for Einstein's and de-Sitter's cosmological models.
2. Discuss Einstein's model of universe and compare it with actual universe.
3. Derive de-sitter's model of the universe and discuss its physical propertier.
4. Describe the three possibilities of a static model of the universe and bring out the similarly and difference between them.
5. Obtain the line element for Robertson Walker non static cosmological model. Show how this model reveals that universe is expanding.

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