M305

Institute of Distance and Open Learning

Gauhati University

M.A./M.Sc. in Mathematics

Semester 3

Paper V

Special Theory of Relativity

Contents:

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Unit-1

1.1 Introduction

The theory of relativity consists of two parts : The special (or Restricted) Theory of Relativity and the General Theory of Relativity. The Special Theory was presented by Einstein in 1905 and the General Theory in 1916. The Special Theory of Relativity had its origin in the development of eletrodynamics. The General Theory of Relativity is the relativistic theory of gravitation.

The Special theory deals only with objects or systems which are either moving at constant velocity with respect to one another (unaccelerated systems) or which are not moving at all (with a constant velocity of zero). The General theory treats of objects or systems which are speeding up or slowing down with respect to one another (accelerated systems). The Special theory is really a particular case of the General theory, since systems moving with constant velocity can be thought of as having an acceleration zero.

1.2 Absolute Motion :

A frame of reference which is supposed to be at absolute rest (i.e. at rest for all times) is called an absolute frame of reference. Any motion relative to such frame of reference is called absolute motion in which time is also considered to be absolute.

The Newtonian motion is based on the idea of absolute frame of reference and time and so it is called absolute motion.

1.3 Law of inertia :

It is stated as "Every body should be at rest or of uniform motion unless it is applied by some external forces to change that state of rest."

1.4 Inertial frame of reference and non-inertial frames:

According to Newton's first law, "A body at rest remains at rest and a body in motion continues with steady speed in a straight line until an external force is applied on the body. This law may also be regarded as the definition of force, i.e. 'Force is the source by which the state of a body whether in motion or at rest may be changed.' Now we know motion of a body has no meaning unless it is described w.r. to some well defined co-ordinate system or frame of reference w.r to which the velocity of a body is measured, i.e. we must choose a meaningful co-ordinate system by which the motion of a body may be described. Newton introduced the idea of "absolute space." In any case a frame of reference must be chosen in such a way that the law of nature may become fundamentally simpler when expressed in terms of such frames of reference.

There are generally two types of reference system :

1. The frame w.r. to which an unaccelerated body is unaccelerated. This also includes the state of rest.

2. The frame w.r. to which an unaccelerated body is accelerated.

Inertial frames

The frame w.r. to which an unaccelerated body appears unaccelenation are inertial frames. In other words the frames which are at rest or in uniform translatory motion relative to one another are inertial frames :

Let us consider any co-ordinate system relative to which a body in motion has co-ordinates (x, y, z). The co-ordinate of the body relative to the assume co-ordinate system are functions of time, so that Newton's first law can be stated in mathematical form, since the body is not being acted by a force.

$$\text{Thus,} \quad \frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=0, \quad \frac{d^2z}{dt^2}=0$$

$$\text{which give} \quad \frac{dx}{dt}=u_x, \quad \frac{dy}{dt}=u_y, \quad \frac{dz}{dt}=u_z$$

where u_x, u_y, and u_z are three components of velocity in x, y, z direction respectively. From these equations the components of the velocity are constant, that is we may say that without the application of an external force, a body in motion continues to be in motion with uniform velocity in a straight line. Which is Newton's first law.

Thus we may always choose a frame of reference or co-ordinate system w.r. to which the body is at rest or in uniform motion. Thus when a body is not subjected to any frame, there exists a frame of reference w.r. to which the body is at rest or moving with constant linear velocity i.e. w.r. to which the body is unacceterated. Such a frame of reference or co-ordinate system is called inertial frame. In brief we may say "An inertial frame is one which Newton's first law is true." Or an unaccelerated frame is inertial frame.

Non-inertial frames

The frames relative to which an unaccelerated body appears accelerated are called non-inertial frames. In other words the accelerated frames are non inertial.

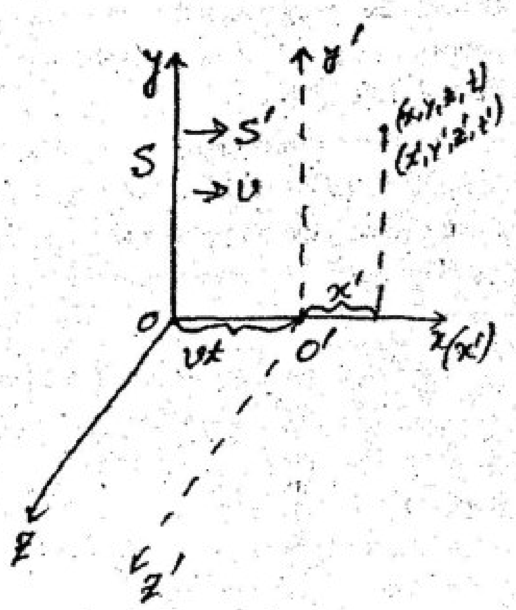
Experiments give an inference that Newton's frame of reference fixed in stars in an inertial frame. A co-ordinate system fixed in earth is not an inertial frame since the earth rotates about its axis and also about the sun. In fact anything which is capable of turning is not an inertial frame. Since though no forces act on the body, but it is neither at rest nor moving in a straight line with constant speed w.r. to such frame of reference.

1.5 Galilean frame of reference :

Let S and S′ be two inertial frames of reference where S′ moves with constant speed. 'v' in the direction of x-axis relative to S.

In this case, the two origin O and O′ coincide at $t = t' = 0$.

Let (x, y, z, t) be the position of a particle with respect to S-frame and (x′, y′, z′, t′) be the position of the same particle w.r.t. S′-frame.



Since the frame S′-moves with constant speed v in the direction of x-axis, therefore $x' = x-vt$, $y' = y$, $z' = z$.

But $t = t'$ (absolute assumption of time)

$x' = x-vt, \qquad y' = y, \qquad z' = z, \qquad t' = t$

is called the Galileas transformation. It is of course, the essential transformation for Newtonian mechanics.

If we consider the motion of the frame S′ with constant speed $v(u^1, u^2, u^3)$ in the direction of a straight line (other than x-axis) then the Galilean transformation can be expressed as

$x' = x-u^1t$, $y' = y-u^2t$, $z' = z-u^3t$, $t' = t$ (absolute time)

This can be thrown to the form–

$$x'^i = x^i - u^i t, \; (x^1, x^2, x^3) \equiv (x, y, z)$$

$$t' = t$$

Differentiating with respect to 't'

$$\frac{dx'^i}{dt'} = \frac{dx^i}{dt} - u^i$$

$$v'^i = v^i - u^i \qquad \text{(A)}$$

Again differentiating w.r.t. 't' we get

$$\frac{dv'^i}{dt'} = \frac{dv^i}{dt} - 0$$

$$f'^i = f^i \qquad \because u^i \text{ is constant.} \qquad \text{(B)}$$

From relation (A) and (B) it is seen that the two velocities are different w.r.t. two inertial frames whereas two acceleration are same w.r.t. two inertial frames. This suggests any mathematical or scientific law involving acceleration will be invariant w.r.t. inertial frames but those laws involving velocity will not be invariant w.r.t. inertial frames. It will be seen later on that this contradiction of invariancy with acceleration and velocity will create problems in mechanics. This will of course be the stepping stone for relativistic mechanic.

1.6 Galilio's Principle of relativity :

All laws of mechanics are invariant w.r.t. observers moving in relative uniform motion (i.e. w.r.t. inertial frames). This principle of relativity was ennuntiated by Galileo before the invertion of the concept of light. But with the emergence of the concept of light two different branches of science viz. electrodynamics and optics developed. Most of the laws of optics and some of the laws of electrodynamics are governed by the velocity of light c.

These laws appear not to be invariant ($\because\ v'^i = v^i - u^i$) w.r.t. inertial frames. This shows that Galileo's principle of Relativity is not applicable to Electrodynamical and optical laws unless c is constant. But we can prove with the help of labortory experiment or with the help of astronomical phenomena, that the speed of light c is a constant quantity. This is proved by the followings :

(1) From the observation of light coming from the binary stars :

Out of two stars in space forming binary lights are observed to come with the help of high power telescope. One of the two stars (once being at the same distance from the earth) goes away from the earth while the other, approaches the earth in certain instant. If lights are observed from the earth in such a stage, they are found to reach the earth simultaneously. Had the speed of light 'c' been dependent on the motion of the source (one star is called the companion and the other is primary), then the light coming from the approaching star should have reached the earth earlier than the light coming from the going away star. But in reality lights from both the stars reach the earth at the same time. This definitely supports that c is not affected at all by the motion of the source. Hence, c is a constant quantity.

(2) Fizeu's experiment :

In a labortary set-up Fizeu's tried to observe fringes of light allowing it to pass through the direction of flowing water and against it, but in reality, he could not observe it (fringes of light). This proves that the speed of light is not at all affected by the medium like the flowing water. Hence 'c' is constant.

(3) Michelson-Morley experiment :

Michelson and Morley tried to establish the absolute motion of the earth relative to the hypothetical fixed ether medium with the help of a labortary set-up with mirrors on the earth.

In their labortary set up, they tried to observe fringes of light so as to determine the absolute motion of earth. But they could not observe the fringes of light. This definitely rejects the concept of absolute motion. The failure of Michelsion-Morley experiment rather supports the constancy of speed of light. Because in the set-up, lights are received along the direction of transformation and against it which could not produce fringes at the receiving position. Hence velocity of light is ultimately found to be constant. But if we want to generalise Galileo's principle of relativity to include Electrodynamical and optical laws then speed of light will come out not to be constant. This assumption of one contradicts the other, but both of them are correct. It was only Einstein who could give a compromise formula assuming the following two as postulates of special theory of Relativity subsequently changing the absolute concept of space and time.

(1) All laws of physics (excluding gravity) must be invariant w.r.t. observers moving in relative uniform motion.

(2) The speed of light c (in vacuum) is constant w.r.t. observers moving in relative uniform motion.

1.7 Absolute concept of space and time :

If (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) be the coordinates of two points w.r.t. an inertial frame of reference then by Galilean transformation, we have,

$$\left.\begin{array}{llll} x_1' = x_1 - vt_1, & y_1' = y_1, & z_1' = z_1, & t_1' = t_1 \\ x_2' = x_2 - vt_2, & y_2' = y_2, & z_2' = z_2, & t_2' = t_2 \end{array}\right] \qquad (1)$$

$$d'^2 = (x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2$$

where d' is the distance between the two points considered in S' frame. Using (1) we get,

$$d'^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$= d^2 \qquad (\because t_1 = t_2)$$

$$\Rightarrow d' = d$$

This shows that the distance between the two points is the same. Otherwise, space which is the composition of length or distance, is the same for both the observers. Also we have already assumed that $t = t'$. Hence space and time are absolute concepts in Newtonian or classical mechanics in which Galilean transformation is the essential transformation. Due to this Galilean transformation, we have got

$$v'^i = v^i - u^i \qquad (\text{at } t = t')$$

which has contradicted the invariancy of the speed of light c.

This idea was the bone of contention before Einstein proposed the two postulates of Special Theory of Relativity. He has rejected the absolute concepts of space and time other than the Galilean transformation and search for new transformation other than Galilean transformation on the basis of two postulates. Hence, motion is not absolute rather it is relative.

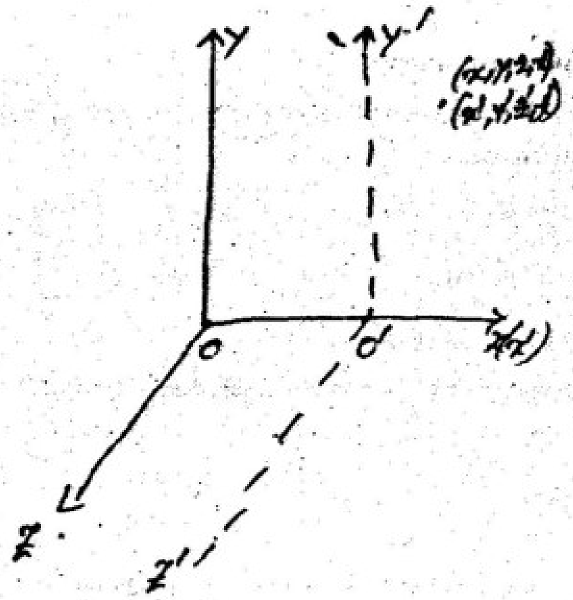
1.8 Lorentz Transformation :

Let Oxyz be an inertial frame S and O′x′y′z′ be another frame S′ which moves with constant speed 'v' in the direction of x-axis relative to S-frame; the two origins are at the same position at

$$t = t' = 0$$

Let p(x, y, z, t) be the coordinates of a point event w.r.t. S-frame at (x′, y′, z′, t′) be the coordinates of the same point w.r.t. S′-frame.

Now the point $x' = 0$ (i.e. the origin of S′-frame) is identical to the point $x = vt$ or $x - vt = 0$ in the language of S-frame.



$$\therefore \frac{x'}{x - vt} = \alpha$$

$$\Rightarrow x' = \alpha(x - vt) \qquad (1)$$

where α is some constant to be determined and is function of 'v' since there are no changes in y and z-direction therefore due to symmetry

$$y = y' \text{ and } z = z' \qquad (2)$$

From the observation of (1) and (2) and remembering that the speed of light is a constant quantity, we can take

$$t' = \beta t + \gamma x \qquad (3)$$

where β and γ are constants but function of v and they are to be determined.

Let us consider a light signal that emits at the instant of separation of S′-frame from S. Since light wave propagates in the form of a sphere, therefore the wavefront is described as

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \ (r^2) \qquad \text{in S-frame} \qquad (4)$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \ (r'^2) \qquad \text{in S}'\text{-frame} \qquad (5)$$

using (1), (2) and (3) in (5) we get–

$$\alpha^2(x - vt)^2 + y^2 + z^2 - c^2(\beta t + \gamma x)^2 = 0$$

or, $$(\alpha^2 - c^2\gamma^2)x^2 - 2xt\,(\alpha^2 v + \beta\gamma c^2) + y^2 + z^2 + (\alpha^2 v^2 - \beta^2 c^2)t^2 = 0$$

which must be identical to (4)

$$\therefore\ \alpha^2 - c^2\gamma^2 = 1 \qquad (6)$$

$$\alpha^2 v + \beta\gamma c^2 = 0 \qquad (7)$$

$$\alpha^2 v^2 - \beta^2 c^2 = -c^2 \qquad (8)$$

(6) × v – (7) gives

$$-c^2\gamma^2 v - \beta\gamma c^2 = v \qquad (9)$$

(8) – (6) × v^2 gives

$$c^2v^2\gamma^2 - \beta^2c^2 = -c^2 - v^2 \qquad (10)$$

Also (8) – (7) × v gives

$$-\beta^2c^2 - \beta\gamma v c^2 = -c^2$$

or $$\beta^2 + \beta\gamma v = 1$$

or $$\gamma = \frac{1-\beta^2}{\beta v}$$

Putting this value of γ in (10) we get

$$c^2v^2 \times \frac{(1-\beta^2)^2}{\beta^2 v^2} - \beta^2c^2 = -c^2 - v^2$$

or $$c^2(1 - 2\beta^2 + \beta^4) - \beta^4c^2 = -\beta^2c^2 - \beta^2v^2$$

or $$c^2 - 2\beta^2c^2 + \beta^2c^2 = -\beta^2v^2$$

or $$\beta^2(c^2 - v^2) = c^2$$

or $$\beta^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - \frac{v^2}{c^2}}$$

or $$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting the value of β^2 in (8) we get

$$\alpha^2 v^2 - c^2 \frac{c^2}{c^2 - v^2} = -c^2$$

$$\Rightarrow \alpha^2 v^2 = -c^2 + \frac{c^4}{c^2 - v^2} = \frac{c^2 v^2}{c^2 - v^2}$$

$$\therefore \alpha^2 = \frac{c^2}{c^2 - v^2} = \beta^2$$

Hence $$\alpha = \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

From (7) $$\beta\gamma c^2 = -\alpha^2 v$$

or, $$\gamma = -\frac{\alpha v}{c^2} \qquad (\because \alpha = \beta)$$

Hence $x' = \alpha(x - vt)$, $y' = y$, $z' = z$

$$t' = \alpha t - \frac{\alpha v}{c^2} x = \alpha\left(t - \frac{vx}{c^2}\right)$$

which is the Lorentz transformation

Case (1) : If $v << c$ then $\frac{v^2}{c^2} \to 0$

$$\therefore \alpha = \beta \qquad \text{(i)}$$

In this case, the Lorentz transformation takes the form

$x' = x - vt$, $y' = y$, $z' = z$ and $t' = t$.

which is the Galilean transformation

Case (2) : Now from Lorentz transformation

$$D = \begin{vmatrix} \beta & 0 & 0 & -\beta v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-\beta v}{c^2} & 0 & 0 & \beta \end{vmatrix} = 1 \begin{vmatrix} \beta & 0 & -\beta v \\ 0 & 1 & 0 \\ \frac{-\beta v}{c^2} & 0 & \beta \end{vmatrix}$$

$$= 1 \times \beta^2 - \beta^2 . \frac{v^2}{c^2} = \beta^2\left(1 - \frac{v^2}{c^2}\right)$$

$$= 1 \neq 0 \qquad \left(\because \beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \right)$$

$\therefore$ **The inverse Lorentz transformation exists.**

$\therefore$ **From** $x' = \beta(x - vt)$ **we get**

$$\frac{x'}{\beta} = x - vt \qquad (ii)$$

From $t' = \beta\left(t - \frac{vx}{c^2}\right)$

$$\Rightarrow \frac{t'}{\beta} = t - \frac{vx}{c^2}$$

or, $$\Rightarrow t = \frac{t'}{\beta} + \frac{vx}{c^2} \qquad (iii)$$

Using (iii) in (ii)

$$\frac{x'}{\beta} = x - v \times \left(\frac{t'}{\beta} + \frac{vx}{c^2}\right)$$

or, $$\frac{x'}{\beta} + \frac{v}{\beta}t' = x\left(1 - \frac{v^2}{c^2}\right) = \frac{x}{\beta^2}$$

or, $$\frac{1}{\beta}(x' + vt') = \frac{x}{\beta^2} \qquad \left(\because \beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \right)$$

or, $x = \beta(x' + vt')$

Similarly $$t = \beta\left(t' + \frac{vx'}{c^2}\right)$$

Also $y = y'$, $z = z'$.

$\therefore$ **Inverse Lorentz transformation is**

$$x = \beta(x' + vt')$$

$$y = y', \qquad z = z'$$

$$t = \beta\left(t' + \frac{vx'}{c^2}\right)$$

This shows that the inverse Lorentz transformation is recoverable from Lorentz transformation replacing v by −v. It can be concluded from the basic idea that the frame S is moving with velocity '−v' w.r.t. 'S''.

1.9 Lorent transformation as a rotation.

We know from elementary co-ordinate geometry that by rotation of x-axis and y-axis through an angle θ to x′-axis and y′-axis respectively, we get the following transformation

$$\left.\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z \\ t' &= t \end{aligned}\right\} \qquad (1)$$

This rotation of axes takes place in the xy-plane about the z-axis.

If we apply the transformation of equation (1) to the rotation of x-axis and (ict) axis through an imaginary angle $\theta = i\phi$ in the complex x-ict plane, then we obtain.

$$\begin{aligned} x' &= x\cos(i\phi) + ict\sin(i\phi) \\ &= x\cosh\phi + i(ict)\sin h\phi \\ &= x\cosh\phi - ct\sinh\phi \\ y' &= y \\ z' &= z \end{aligned}$$

$$\left.\begin{aligned} ict' &= (ict)\cos(i\phi) - x\sin(i\phi) \\ &= ict\cosh\phi - ix\sinh\phi \\ \text{i.e.} \quad ct' &= ct\cosh\phi - x\sinh\phi \end{aligned}\right\} \qquad (2)$$

It can be easily seen that if

$$\left.\begin{aligned} \cosh\phi &= \frac{1}{\sqrt{1 - v^2/c^2}} \\ \text{and} \quad \sinh\phi &= \frac{v/c}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \qquad (3)$$

then equation (2) reduces to the Einstein-Lorentz transformation. This transformation is called the Lorentz rotation of axes.

1.10 Lorentz transformation as a group

Consider three frames of reference S, S′, S″, where S′ is moving with velocity v relative to S along x-axis and S″ is moving with velocity v′ relative to S′ along x-axis. By Lorentz transformation equations the frames S and S′ can be related as

$$x' = \beta(x - vt), \quad y' = y, \quad z' = z, \quad t' = \beta\left(t - \frac{vx}{c^2}\right) \qquad (1)$$

where $\beta = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

Similarly S′ and S″ can be related as $x'' = \beta'(x' - v't')$, $y'' = y'$, $z'' = z'$,

$$t'' = \beta'\left(t' - \frac{v'x'}{c^2}\right)$$

where $\beta' = \dfrac{1}{\sqrt{1 - v'^2/c^2}}$

Let v″ be the resultant of v and v′ so that

$$v'' = \frac{v + v'}{1 + \frac{vv'}{c^2}} \qquad (3)$$

v″ is the velocity of the system S″ relative to S.

writing $\beta'' = \dfrac{1}{\sqrt{1 - \frac{v''^2}{c^2}}}$

We are to show

$$x'' = \beta''(x - v''t), \quad y'' = y, \quad z'' = z, \quad t'' = \beta''\left(t - \frac{v''x}{c^2}\right) \qquad (4)$$

$$\frac{1}{\beta''^2} = 1 - \frac{v''^2}{c^2}$$

$$=1-\frac{1}{c^2}\left(\frac{v+v'}{1+\frac{vv'}{c^2}}\right)^2=\frac{c^2\left(1+\frac{vv'}{c^2}\right)^2-(v+v')^2}{c^2\left(1+\frac{vv'}{c^2}\right)^2}$$

$$=\frac{c^2\left(1+\frac{v^2v'^2}{c^4}+2\frac{vv'}{c^2}\right)-\left(v^2+v'^2+2vv'\right)}{c^2\left(1+\frac{vv'}{c^2}\right)^2}$$

$$=\frac{c^2+\frac{v^2v'^2}{c^2}+2vv'-v^2-v'^2-2vv'}{c^2\left(1+\frac{vv'}{c^2}\right)^2}$$

$$=\frac{c^2\left[1+\frac{v^2v'^2}{c^4}-\frac{v^2}{c^2}-\frac{v'^2}{c^2}\right]}{c^2\left(1+\frac{vv'}{c^2}\right)^2}=\frac{\left(1-v^2/c^2\right)\left(1-v'^2/c^2\right)}{\left(1+\frac{vv'}{c^2}\right)^2}$$

So,

$$\beta''=\frac{\left(1+\frac{vv'}{c^2}\right)}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)\left(\sqrt{1-\frac{v'^2}{c^2}}\right)}=\beta\beta'\left(1+\frac{vv'}{c^2}\right)$$

Thus, $$\beta''=\beta\beta'\left(1+\frac{vv'}{c^2}\right) \qquad (5)$$

Writting equation (2) with the help of equation (1).

$$x''=\beta'\left(x'-v't'\right)=\beta'\left[\beta(x-vt)-v'\beta\left(t-\frac{vx}{c^2}\right)\right]$$

$$= \beta\beta'\left[x\left(1+\frac{vv'}{c^2}\right)-t(v+v')\right] = \beta\beta'\left(1+\frac{vv'}{c^2}\right)\left(x-\frac{v+v'}{1+\frac{vv'}{c^2}}t\right)$$

$$= \beta''(x - v''t) \qquad \text{using equ}^n\ (5)$$

$$x'' = \beta''(x - v''t)$$

Again from equation (2) and (1) we get

$$t'' = \beta'\left(t' - \frac{v'x'}{c^2}\right) = \beta'\left[\beta\left(t-\frac{vx}{c^2}\right) - \frac{v'^2}{c^2}\beta(x-vt)\right]$$

$$= \beta\beta'\left[t\left(1+\frac{vv'}{c^2}\right) - \frac{x}{c^2}(v+v')\right] = \beta\beta'\left(1+\frac{vv'}{c^2}\right)\left(t-\frac{v+v'}{1+\frac{vv'}{c^2}}\frac{x}{c^2}\right)$$

$$= \beta''\left(t - \frac{xv''}{c^2}\right) \qquad \text{using equation (3) and (5)}$$

$$y'' = y, \qquad y' = y \Rightarrow y'' = y$$

and $\quad x'' = z, \qquad z' = z \Rightarrow z'' = z$

Thus we have shown that

$$x'' = \beta''(x - v''t), \qquad y'' = y, \qquad z'' = z, \qquad t'' = \beta''\left(t - \frac{v''x}{c^2}\right)$$

Hence Lorentz transformation form a group.

1.11 Postulafes of special theory of relativity.

(1) "The fundamental laws of physics have the same form for all inertial systems, i.e. for all reference system at rest or moving with constant linear velocity relative to one another".

(2) The velocity of light is vacuum is independent of the relative motion of the source and the observer.

These are the two fundamental postulates used in the special theory of relativity. The first postulate is the extension of the conclusion drawn from Newtonian mechanies; since velocity is not absolute, but relative which is a fact drawn from the failure of the experiments to determine the velocity of earth relative to ether.

We know that the speed of light is not constant under Galikan transformations and the first postulate is the conclusion from Newtonian mechanics; thus 2nd postulate is not true according to Galilean transformations. Actually this is true since the velocity of light calculated by any mean is a constant. Thus the 2nd postulate is very important and only this postulate is responsible to differentiate the classical theory and Einstein's theory of relativity. According to Einstein the theory of relativity is applicable to laws of optics. Thus for the constancy of velocity of light we have to introduce the new transformation equations which fulfil the following requirements :

1. The speed of light c must have the same value in every inertial frame.

2. The transformation must be linear and for low speeds for $v << c$ they should approach the Galilean transformations.

3. They should not be based an absolute time and absolute space.

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UNIT 2

2.1 Introduction :

In special theory of relativity if a particle is moving with velocity of light c, it will be observed c in all inertial frames whatever their relative speed is. It follows that "the velocity of light is an absolute constant independent of the motion of the reference system". But there are many physical phenomenon and laboratory experiment supported that the velocity of light is an upper limit of the velocity of a particle with which a particle can move in nature.

This fact is also expressed by saying that it is impossible to send out signals with a velocity greater than the speed of light.

2.2 Relativistic addition law of velocity or composition law of velocities :

Let S and S′ be two inertial frames of reference where S′ moves with constant speed 'v' in the direction of x-axis relative to S-frame, the two origins being at the same position at $t = t' = 0$.

∴ By Lorentz transformation

$$x' = \beta(x - vt), \quad y' = y, \quad z' = z,$$

$$t' = \beta\left(t - \frac{vx}{c^2}\right)$$

or the inverse Lorentz transformation

$$x = \beta(x' + vt'), \quad y = y', \quad z = z',$$

$$t = \beta\left(t' + \frac{vx'}{c^2}\right) \qquad (1)$$

Let (u_x, u_y, u_z) and (u_x', u_y', u_z') be the velocities of a point event w.r.t. S and S′ frame respectively.

$$u_x = \frac{dx}{dt}, u_y = \frac{dy}{dt}, u_z = \frac{dz}{dt}$$

and $$u_x' = \frac{dx'}{dt}, u_y' = \frac{dy'}{dt}, u_z' = \frac{dz'}{dt}$$

From (1)

$$dx = \beta(dx' + vdt'), \quad dy = dy', \quad dz = dz'$$

$$dt = \beta\left(dt' + \frac{v}{c^2}dx'\right)$$

∴ since

$$\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{is constant}$$

$$\therefore u_x = \frac{dx}{dt} = \frac{\beta(dx' + vdt')}{\beta\left(dt' + \frac{v}{c^2}dx'\right)} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}$$

Also,

$$u_y = \frac{dy}{dt} = \frac{dy'}{\beta\left(dt' + \frac{vdx'}{c^2}\right)} = \frac{u_y'}{\beta\left(1 + \frac{v}{c^2}u_x'\right)}$$

Similarly,

$$u_z = \frac{u_z'}{\beta\left(1 + \frac{vu_x'}{c^2}\right)}$$

If we take $u_x' = u$, $u_y' = 0$ and $u_z' = 0$ i.e. if we assume that the point event is moving with velocity 'u' but also in the x-direction only, then

$$(u_x =)\, V = \frac{u + v}{1 + \frac{uv}{c^2}}$$

Which is called the composition law or relativistic addition law of two velocities.

If u < < c, v < < c then

$$V = u + v \qquad \text{i.e.} \qquad \frac{uv}{c^2} \to 0$$

which is the classical addition law of two velocities in relativistic sense.

Deduction : (1) By the addition law of velocities $V = \frac{u + v}{1 + \frac{uv}{c^2}}$

Let us take u = c, then

$$V = \frac{c + v}{1 + \frac{v}{c}} = c$$

Deduction : (2)

The addition of two velocities each of which is less than c is also less than c.

Let u and v be two velocities each of which is less than c, so that

$$u = c - \lambda, \qquad v = c - \mu; \qquad \lambda, \mu > 0$$

∴ From the addition law $V = \dfrac{u+v}{1+\dfrac{uv}{c^2}}$

we get

$$V = \frac{(c-\lambda)+(c-\mu)}{1+\dfrac{(c-\lambda)(c-\mu)}{c^2}}$$

$$= c^2 \frac{2c-(\lambda+\mu)}{c^2+\{c^2-(\lambda+\mu)c+\lambda\mu\}}$$

$$= c^2 \frac{2c-(\lambda+\mu)}{2c^2-(\lambda+\mu)c+\lambda\mu}$$

$$\therefore V = c\frac{2c-(\lambda+\mu)}{2c-(\lambda+\mu)+\dfrac{\lambda\mu}{c}}$$

∴ $V < c.1$ since $\lambda, \mu > 0$. i.e. $V < c$.

Deduction (3)

It is impossible to send a signal with velocity greater than the velocity of light c.

Let if possible a signal is sent with velocity 'w' greater than the velocity of light 'c' w.r.t. an inertial frame S.

Let AB = l be the distance to be covered by the signal. Now if v' be the velocity of the signal w.r.t. the frame S' which moves with constant speed V, in the direction of x-axis relative to S, then

$$V' = \frac{w-v}{1-\dfrac{wv}{c^2}}$$

∴ If t' be the time required by the signal to cover the distance AB w.r.t. S'-frame then

$$t' = \frac{l}{v'} = \frac{l}{\dfrac{w-v}{1-\dfrac{wv}{c^2}}} = \frac{l\left(1-\dfrac{wv}{c^2}\right)}{w-v} \qquad \because\ w > c \qquad (1)$$

$$\therefore\ w = c + \lambda \qquad (\lambda > 0)$$

and $\quad v < c \qquad \therefore\ v = c - \mu\ (\mu > 0)$

$$\therefore\ w - v > 0.$$

Now, $\quad wv = (c + \lambda)(c - \mu)$

$$= c^2 + (\lambda - \mu)c - \lambda\mu$$

$$\therefore\ wv > c^2 \quad \text{if} \quad (\lambda - \mu)c - \lambda\mu > 0$$

$$\text{or,} \quad \frac{wv}{c^2} > 1 \quad \text{if} \quad (\lambda - \mu) > \frac{\lambda\mu}{c}$$

$\therefore$ Subject to this condition $\quad t' =$ negative quantity $\qquad$ (from (1))

This shows that the signal with velocity w (> c) will reach the position of B before it is signaled. Otherwise the effect of an action will be perceived before the commencement of the action. This can be made always possible suitably selecting λ and μ i.e. w and υ. Hence it is impossible to send a signal with a velocity exceeding the velocity of light.

2.3 Consequences of Lorentz Transformation.

(1) Lorentz-Fitzerald Contraction :

Let S and S′ be two inertial frames of reference where S′ moves with constant speed 'v' in the direction of x-axis relative to S-frame. The two origins O and O′ coincide at $t = t' = 0$.

Case I : Let us consider a rod AB of length 'l' which is placed parallel to the direction of y-axis.

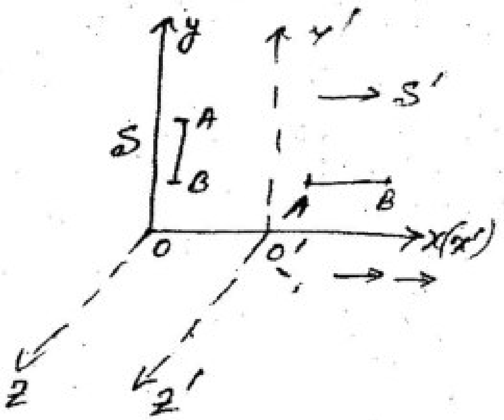
Let y_1 and y_2 be the y-coordinates of A and B respectively of the rod w.r.t. S-frame.

$$\therefore\ (AB =)\ l = y_1 - y_2.$$

If y_1' and y_2' be the y-coordinates of the ends A and B respectively w.r.t. S′-frame then

$$l' = y_1' - y_2' = y_1 - y_2 = l$$

This shows that the length of the rod when placed perpendicular to the direction of motion remains unchanged w.r.t. both the observers in S and S′-frame.



Case II : Let us place the rod AB = l in S-frame parallel to the direction of motion. i.e. parallel to x-axis.

Let x_1 and x_2 be the coordinates of the end points A and B respectively of the rod w.r.t. S-frame.

$$\therefore l = x_2 - x_1 \qquad \text{(i)}$$

Let x_1' and x_2' be the coordinates of A and B respectively of the end points measured at the same time in S′-frame

$$\therefore l' = x_2' - x_1' \qquad \text{(ii)}$$

From (i)

$$l = \beta(x_2' + vt_2') - \beta(x_1' + vt_1')$$

$$= \beta(x_2' - x_1')$$

$$\Rightarrow l = \beta l'$$

$$\because \beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \left(1-\frac{v^2}{c^2}\right)$$

$$= 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\ldots$$

$$\therefore \beta > 1$$

$$\therefore l > l'$$

i.e. the length of the rod is found to be contracted in the moving frame S′. This is known as the Lorentz Fitzerald contraction.

If the rod is placed in S′ frame and observation is made from S frame then we get

$$l' = \beta l \quad \text{i.e.} \quad l' > l \quad \text{because} \quad \beta > 1$$

Thus it can be concluded that "every body appears to be greatest when it is at rest relative to the observer otherwise it is found to be contracted in the direction of motion when it(body) is in motion relative to the observer."

(2) Time Dilation :

Let S and S′ be two inertial frames, where S′ moves with constant speed v in the direction of x axis relative to the S frame and the two origins coincide at $t = t' = 0$.

Let a clock be placed in S frame. Let the two times recorded by this clock for an event taking place at the same position in S frame be t_1 and t_2.

Let t_1' and t_2' be the corresponding times of record of the event measured in the S′ frame. By Lorentz transformation

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right) \text{ and } t_2' = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

$$\therefore \quad t_2' - t_1' = \beta(t_2 - t_1) - \beta\frac{v}{c^2}(x_2 - x_1)$$

$$= \beta(t_2 - t_1) \qquad [\because \quad x_1 = x_2]$$

$$t_2' - t_1' = \beta(t_2 - t_1) \Rightarrow \Delta t' = \beta\Delta t$$

$$\because \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \frac{v^2}{c^2}\right)^{\frac{-1}{2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots..$$

$$\therefore \beta > 1$$

So, $\Delta t' > \Delta t$

This shows that the time interval in S' is found to be greater than that of S i.e. the time interval is found to be lengthened or dialated. This is known as time dialation.

On the otherhand if the clock is placed in S'-frame and the time interval is observed from S-frame, then we could get

$$\therefore \Delta t = \beta\Delta t' \qquad \text{i.e.} \qquad \Delta t > \Delta t'$$

Apparently it shows that the time interval in S' is dialated. It appears to be a contradiction. For in the 1st case it is S's clock that goes slow. This is known as clock paradox. Thus every clock goes slow when it is in motion relative to the observer.

3. Simultaneity of events :

Let two events occur simultaneously at two positions x_1 and x_2 w.r.t. the observer at the origin O and an inertial frame S.

Let x_1' and x_2' be the positions of the two events w.r.t. another observer at O' in a frame S' which moves with constant speed 'v' in the direction of 'x' axis relative to S. The two observers at O and O' are at the same position at $t = t' = 0$.

$\therefore$ By Lorentz transformation

$$x_1' = \beta(x_1 - vt_1), \qquad x_2' = \beta(x_2 - vt_2)$$

and

$$t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right), \qquad t_2' = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

$$\therefore t_2' - t_1' = \beta\frac{v}{c^2}(x_1 - x_2) \qquad (\because t_1 = t_2) \qquad \because t_2' - t_1' \neq 0 \qquad \therefore t_2' \neq t_1'$$

Hence, the events which are simultaneous w.r.t. one observer, is not simultaneous w.r.t. other observer moving in relative uniform motion.

Thus the simultaneity of events w.r.t. one observer does not imply simultaneity of the events w.r.t. another observer moving in relative inertial frames.

2.4. Transformation of acceleration–

Let S and S′ be two systems, S′ moving with velocity v relative to S along positive direction of x-axis. If there is a particle moving with velocity u′ relative to frame of reference S′, then components of u′ as observe from system S are according to law of composition of velocities

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}}, \quad u_y = \frac{u_y'\sqrt{1-\alpha^2}}{1 + \frac{u_x' v}{c^2}} \quad \text{and} \quad u_z = \frac{u_z'\sqrt{1-\alpha^2}}{1 + \frac{u_x' v}{c^2}} \qquad (1)$$

Now if f′ is the acceleration of a particle in frame S′; its components along the three axes being f_x', f_y', f_z' then we have,

$$f' = if_x' + jf'_y + kf'_z$$

where i, j, k being unit vectors along their respective axes.

We have to find the acceleration f relative to frame S. Let the component of f along three axis be f_x, f_y and f_z then

$$f = if_x + jf_y + kf_z$$

Thus $$f_x = \frac{du_x}{dt}, \quad f_y = \frac{du_y}{dt} \quad and \quad f_z = \frac{du_z}{dt}.$$

Now for the purpose taking differentiation of (1), we have,

$$du_x = \frac{du_x'\left(1 + \frac{vu_x'}{c^2}\right) - (u_x' + v)\frac{vdu_x'}{c^2}}{\left(1 + \frac{vu_x'}{c^2}\right)^2}$$

$$= \frac{du_x' + \frac{v}{c^2}u_x'du_x' - \frac{v}{c^2}u_x'du_x' - \frac{v^2}{c^2}du_x'}{\left(1 + \frac{vu_x'}{c^2}\right)^2}$$

$$= \frac{\left(1-\frac{v^2}{c^2}\right)du'_x}{\left(1+\frac{v}{c^2}u'_x\right)^2} \qquad (2)$$

$$du = \frac{du'_y\sqrt{1-\alpha^2}\left(1+\frac{v}{c^2}u'_x\right)-\frac{v}{c^2}du'_x u'_y\sqrt{1-\alpha^2}}{\left(1+\frac{v}{c^2}u'_x\right)^2} \qquad \text{where } \alpha = \frac{v}{c}$$

$$= \frac{du'_y + \frac{v}{c^2}u'_x\,du'_y - \frac{v}{c^2}du'_y u'_y}{\left(1+\frac{v}{c^2}u'_x\right)^2}$$

$$du_y = \sqrt{1-\alpha^2}\,\frac{\left(1+\frac{v}{c^2}U'_x\right)du'_y - \frac{v}{c^2}u'_y du'_x}{\left(1+\frac{v}{c^2}u'_x\right)^2} \qquad (3)$$

and similarly

$$du_z = \sqrt{1-\alpha^2}\,\frac{\left(1+\frac{v}{c^2}u'_x\right)du'_z - \frac{v}{c^2}u'_z du'_x}{\left(1+\frac{v}{c^2}u'_x\right)^2} \qquad (4)$$

Also from inverse Lorentz transformation we have,

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\alpha^2}}$$

Taking differentials, we have,

$$dt = \frac{dt' + \frac{vdx'}{c^2}}{\sqrt{1-\alpha^2}}$$

$$\frac{dt}{dt'} = \frac{1+\frac{v}{c^2}u_x'}{\sqrt{1-\alpha^2}} \qquad (5)$$

Thus from (1), we have

$$f_x = \frac{du_x}{dt} = \frac{\left(1-\frac{v^2}{c^2}\right)\frac{du_x'}{dt'}\frac{dt'}{dt}}{\left(1+\frac{v}{c^2}u_x'\right)^2}$$

$$= \frac{\left(1-\frac{v^2}{c^2}\right)f_x'\sqrt{1-\alpha^2}}{\left(1+\frac{v}{c^2}u_x'\right)^2\left(1+\frac{v}{c^2}u_x'\right)} \qquad \text{using (5)}$$

$$\therefore f_x = \frac{\left(1-\alpha^2\right)^{3/2} f_x'}{\left(1+\frac{v}{c^2}u_x'\right)^3} \qquad (6)$$

Also from equation (3), we have,

$$f_y = \frac{du_y}{dt} = \sqrt{1-\alpha^2}\,\frac{\left(1+\frac{v}{c^2}u_x'\right)\frac{du_y}{dt'}\cdot\frac{dt}{dt} - \frac{v}{c^2}u_y'\frac{du_x'}{dt'}\frac{dt'}{dt}}{\left(1+\frac{v}{c^2}u_x'\right)^2}$$

$$= \sqrt{1-\alpha^2}\,\frac{\left(1+\frac{v}{c^2}u_x'\right)f_y' - \frac{v}{c^2}u_y'f_x'}{\left(1+\frac{v}{c^2}u_x'\right)^2}\,\frac{\sqrt{1-\alpha^2}}{\left(1+\frac{v}{c^2}u_y'\right)}$$

$$\therefore f_y = (1-\alpha^2)\left[\frac{f_y'}{\left(1+\frac{v}{c^2}u_x'\right)^2} - \frac{\frac{u}{c^2}u_y'f_x'}{\left(1+\frac{v}{c^2}u_x'\right)^3}\right] \qquad (7)$$

Similarly from equation (4) we have,

$$f_z = \frac{du_z}{dt} = \left(1-\alpha^2\right)\left[\frac{f_z'}{\left(1+\frac{v}{c^2}u_x'\right)^2} - \frac{\frac{v}{c^2}u_z' f_x'}{\left(1+\frac{v}{c^2}u_x'\right)^3}\right]$$

The equations (6), (7) and (8) represents the components of acceleration f as viewed from system S in terms of components f_x', f_y' and f_z' of f' observed in system S'. From these equations it is clear that acceleration which is constant in one frame of reference is not generally constant in another frame of reference, because the components of acceleration in system S also contain the components of velocity along with those of acceleration.

Now consider the particular case where the particle under consideration in S' is at rest relative to frame S. Then since u' = 0.

$\therefore u_x' = 0, u_y' = 0$ and $u_z' = 0$ therefore the equations (6), (7) and (8) take the form,

$$\left.\begin{aligned} f_x &= (1-\alpha^2)^{3/2} f_x' \\ f_y &= (1-\alpha^2)\ f_y' \\ f_z &= (1-\alpha^2)\ f_z' \end{aligned}\right\} \quad (9)$$

From equations (9) it is clear that unlike the Galilean transformations the acceleration is different in the two inertial frames. This difference may be stated due to relatively of space and time.

●●●

Unit -3

3.1 Introduction :

The formulation of Lorentz transformation is based on the requirement that the law of electrodynamics (in particular Maxwell's equations) remain invariant. The electromagnetic disturbance is propagated with speed (3×10^8 m/sec) in free space, independent of the choice of reference system and no signal can travel with a speed greater than this speed c.

3.2 Relativistic mechanics :

Variation of mass : In classical mechanics we consider space and time as obsolute concepts. But in relativistic mechanics, they are not obsolute concepts. At the same time mass is supposed to be conserved and in absence of external forces the momentum remains conserved. But in relativistic mechanics we will see that if the momentum is supposed to be conserved, then mass of the body will be subjected to change with velocity. i.e. mass will be come out not to be constant.

Let m_1u_1 and $m_1'u_1'$ be the momentum of a body w.r.t. two inertial frames S and S′ where S′ moves with constant speed 'v' relative to S-frame but in the direction of x-axis; the two origins being at the same position at $t = t' = 0$.

Let there exists a number of bodies for which mass and momentum are conserved in S-frame i.e. Σm_1 = constant and Σm_1u_1 = constant.

Let

$$\beta_1 = \frac{1}{\sqrt{1-\frac{u_1^2}{c^2}}}, \qquad \beta_1' = \frac{1}{\sqrt{1-\frac{u_1'^2}{c^2}}} \qquad \beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Since β and v are constants therefore $\Sigma m_1\beta v$ = constant, $\Sigma m_1u_1\beta$ = constant

$$\therefore \Sigma m_1\beta\,(u_1 - v) = \text{constant} \qquad (1)$$

Again, $$\frac{1}{\beta_1'^2} = \left(1-\frac{u_1'^2}{c^2}\right)$$

But by the addition law of velocities

$$u_1' = \frac{u_1 - v}{1-\frac{u_1 v}{c^2}}$$

$$\therefore \quad \frac{1}{\beta_1'^2} = 1 - \frac{1}{c^2}\left(\frac{u_1 - v}{1 - \frac{u_1 v}{c^2}}\right)^2$$

$$= \frac{c^2\left(1 - \frac{2u_1 v}{c^2} + \frac{u_1^2 v^2}{c^4}\right) - (u_1^2 - 2u_1 v + }{c^2\left(1 - \frac{u_1 v}{c^2}\right)^2}$$

$$= \frac{(c^2 - u_1^2) - v^2\left(1 - \frac{u_1^2}{c^2}\right)}{c^2\left(1 - \frac{u_1 v}{c^2}\right)^2} = \frac{c^2\left(1 - \frac{u_1^2}{c^2}\right) - v^2\left(1 - \frac{u_1^2}{c^2}\right)}{c^2\left(1 - \frac{u_1 v}{c^2}\right)^2}$$

$$\therefore \quad \frac{1}{\beta_1'^2} = \frac{(c^2 - v^2)\left(1 - \frac{u_1^2}{c^2}\right)}{c^2\left(1 - \frac{u_1 v}{c^2}\right)^2}$$

or

$$\frac{1}{\beta_1'^2} = \frac{\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u_1^2}{c^2}\right)}{\left(1 - \frac{u_1 v}{c^2}\right)^2} = \frac{1}{\beta^2 \beta_1^2}\frac{1}{\left(1 - \frac{u_1 v}{c^2}\right)^2}$$

$$\Rightarrow \beta_1' = \beta\beta_1\left(1 - \frac{u_1 v}{c^2}\right)^2$$

$$\therefore \quad \frac{\beta_1' u_1'}{\beta_1} = \beta\,(u_1 - v) \qquad \left[\because u_1' = \frac{u_1 v}{1 - \frac{u_1 v}{c^2}}\right]$$

$\therefore$ From (1) $\quad \Sigma m_1 \dfrac{\beta_1' u_1'}{\beta_1} = \text{constant}$ (2)

Let us assume that the conservation of momentum holds in S′-frame also so that

$\Sigma m_1' u_1' = \text{constant}$ (3)

Comparing (2) and (3), we get

$$m_1' = m_1\left(\frac{\beta_1'}{\beta_1}\right)$$

or, $$\frac{m_1'}{\beta_1'} = \frac{m_1}{\beta_1} = \text{a constant} = m_0 \text{ (say)}$$

$$\therefore \quad m_1' = m_0\beta_1' = \frac{m_0}{\sqrt{1-\frac{u_1'^2}{c^2}}}$$

$$m_1 = m_0\beta_1 = \frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

Hence if m is the mass of a body which moves with velocity 'u' then from the above results

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (4)$$

This shows that the mass of a moving body is function of its velocity 'u' and so it is not constant. But if u = 0 then

$$\frac{u^2}{c^2} = 0 \qquad \therefore m = m_0$$

Which is called rest mass of body. So 'm_0' is called rest mass of body. In Newtonian mechanics it is this rest mass of the body 'm_0' which is considered throughout the motion and it is a constant quantity.

If $u << c$, for a motion then also $m \to m_0$, the rest mass. Hence the Newtonian assumption for constant mass is a good approximation for motion attaining velocity not comparable to the velocity of light.

Deduction : If $u \to c$, then $m \to \infty$. i.e. the mass of the body will be infinite if it is possible to impart velocity of light 'c' to the moving body.

3.3 Equivalent of mass and energy OR Relativistic equation of energy

If m be the mass of a body moving with velocity 'υ' (not constant) w.r.t. an observer in an inertial frame S.

If m_0 is its rest mass then, we know that

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \quad dm = m_0\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{\frac{-3}{2}}\left(\frac{-2v}{c^2}dv\right)$$

$$= \frac{m_0 v dv}{c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

or,

$$c^2 dm = \frac{m_0 v dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \qquad (1)$$

Let T be the kinetic energy of the body subjected to the action of the external force F applied for the time interval dt. If during this time interval 'dt' the mass changes from m_0 to $m_o + dm$, then the work done is given by

$$dT = Fdr.$$

'dr' being the displacement due to the applicationof F.

$$= F\frac{dr}{dt}dt = Fv\ dt$$

$$\therefore \quad dT = vd(mv) = vd\left[\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right]$$

$$= m_0 v\left[\frac{\sqrt{1-\frac{v^2}{c^2}}dv + v\frac{\frac{2v^2}{c^2}}{2\sqrt{1-\frac{v^2}{c^2}}}dv}{\sqrt{1-\frac{v^2}{c^2}}}\right]$$

$$dT = m_0 v \frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} dv$$

∴ From (1) and (2), we get $dT = c^2 dm$

$$T = c^2 \int_{m_0}^{m} dm = c^2(m - m_0)$$

$$\Rightarrow T + m_0 c^2 = mc^2$$

$$\Rightarrow E = mc^2$$

where $E = T + m_0 c^2$

= energy of the body due to motion + energy of the body preserved by it while it is at rest.

= total energy of the body.

= energy of motion + energy at rest.

which is called the total energy.

$$\therefore E = mc^2$$

This is known as the relativistic equation of energy or Einstein's equation of energy.

From the appearance of the equation it is observed that mass and energy are equivalent i.e. they are not independent.

3.4 Transformation law of mass :

Let m and m' be the masses of a body moving with velocity $u(u_x, u_y, u_z)$ and $u'(u_x', u_y', u_z')$ w.r.t. S and S' frame respectively where S' moves with constant velocity 'v' w.r.t. S-frame in the direction of x-axis. The two origins are at the same position at $t = t' = 0$.

$$\therefore m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}, \qquad m' = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}} \qquad (1)$$

where $u^2 = u_x^2 + u_y^2 + u_z^2$

$$u'^2 = u_x'^2 + u_y'^2 + u_z'^2$$

$$\therefore \quad \frac{m'}{m}=\frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}}$$

$$\text{or,} \qquad m'=\frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}}\,m$$

Now by addition law of velocity

$$u'_x=\frac{u_x-v}{\left(1-\frac{vu_x}{c^2}\right)}, \qquad u'_y=\frac{u_y}{\beta\left(1-\frac{vu_x}{c^2}\right)} \qquad u'_z=\frac{u_z}{\beta\left(1-\frac{vu_x}{c^2}\right)}$$

$$\therefore \quad 1-\frac{u'^2}{c^2}=1-\frac{1}{c^2}\left(u_x'^2+u_y'^2+u_z'^2\right)$$

$$=1-\frac{1}{c^2}\left[\left(\frac{u_x-v}{1-\frac{vu_x}{c^2}}\right)^2+\frac{u_y^2}{\beta^2\left(1-\frac{vu_x}{c^2}\right)^2}+\frac{u_z^2}{\beta^2\left(1-\frac{vu_x}{c^2}\right)^2}\right]$$

$$=1-\frac{1}{c^2\left(1-\frac{vu_x}{c^2}\right)^2}\left[\left\{\left(u_x^2+v^2-2u_xv\right)+u_y^2\left(1-\frac{v^2}{c^2}\right)+u_z^2\left(1-\frac{v^2}{c^2}\right)\right\}\right]$$

$$=1-\frac{1}{c^2\left(1-\frac{vu_x}{c^2}\right)^2}\left[u^2-2vu_x+v^2-\frac{v^2}{c^2}\left(u_y^2+u_z^2\right)\right]$$

$$=\frac{1}{c^2\left(1-\frac{vu_x}{c^2}\right)^2}\left[c^2\left(1-\frac{2vu_x}{c^2}+\frac{v^2u_x^2}{c^4}\right)-\left\{u^2-2vu_x+v^2-\frac{v^2}{c^2}\left(u_y^2+u_z^2\right)\right\}\right]$$

$$=\frac{1}{c^2\left(1-\frac{vu_x}{c^2}\right)^2}\left[c^2-u^2-v^2+\frac{v^2}{c^2}u^2\right]$$

$$= \frac{1}{c^2\left(1-\frac{vu_x}{c^2}\right)^2}\left[c^2\left(\frac{v^2}{c^2}\right)-u^2\left(1-\frac{v^2}{c^2}\right)\right]$$

$$= \frac{1}{c^2\left(1-\frac{vu_x}{c^2}\right)}c^2\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u^2}{c^2}\right)$$

$$\therefore \sqrt{1-\frac{u'^2}{c^2}} = \frac{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}}{\left(1-\frac{vu_x}{c^2}\right)}$$

or, $$\frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{1-\frac{vu_x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Hence from (1)

$$m' = \frac{\left(1-\frac{vu_x}{c^2}\right)m}{\sqrt{1-\frac{v^2}{c^2}}}$$

which is the transformation law of mass

3.5 Transformation formula for momentum and energy :

Let m and m′ be the masses of a body moving with velocities $u(u_x, u_y, u_z)$ and $u'(u_x', u_y', u_z')$ w.r.t. two inertial frames S and S′ respectively where S′ moves with constant speed v in the direction of x-axis relative to S.

If $p(p_x, p_y, p_z)$ and $p'(p_x', p_y', p_z')$ be the momentum of the body w.r.t. S and S′ frame respectively then

$$p_x' = m'u_x', \qquad p_y' = m'u_y', \qquad p_z' = m'u_z'$$

& $$p_x = mu_x, \qquad p_y = mu_y, \qquad p_z = mu_z.$$

But $$m' = \frac{m\left(1-\frac{vu_x}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (1)$$

and $u_x' = \dfrac{u_x - v}{1 - \dfrac{vu_x}{c^2}}$, $u_y' = \dfrac{u_y}{\beta\left(1 - \dfrac{vu_x}{c^2}\right)}$, $u_z' = \dfrac{u_z}{\beta\left(1 - \dfrac{vu_x}{c^2}\right)}$

$$\therefore \quad p_x' = \frac{m\left(1 - \frac{vu_x}{c^2}\right)}{\left(1 - \frac{v^2}{c^2}\right)} \times \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \qquad \text{(using (1) \& (2))}$$

$$= \frac{mu_x - mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \beta\left(p_x - \frac{E}{c^2}v\right) \qquad \left[\because E = mc^2,\ \therefore m = \frac{E}{c^2}\right]$$

Also $$p_y' = \frac{m\left(1 - \frac{vu_x}{c^2}\right)}{\left(1 - \frac{v^2}{c^2}\right)} \times \frac{u_y}{\beta\left(1 - \frac{vu_x}{c^2}\right)}$$

$$= mu_y = p_y$$

Similarly $p_z' = p_z$

Now $$E' = m'c^2 = \frac{m\left(1 - \frac{vu_x}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}}c^2 = \frac{m'c^2 - mvu_x}{\sqrt{1 - \frac{v^2}{c^2}}} = \beta\left(E - \frac{v}{x}\right)$$

Thus $p_x' = m'u_x' = \beta\left(p_x - \dfrac{Ev}{c^2}\right)$, $p_y' = p_y$, $p_z' = p_z$

$$E' = \beta(E - vp_x)$$

Which are the transformation laws of momentum and energy.

It is observed from these results that the transformation laws of momentum and energy are equivalent to those of Lorentz transformation as if, 't' is replaced by $\dfrac{E}{c^2}$ i.e. $t = \dfrac{E}{c^2}$

3.6 Formulate the energy momentum vector of space-time continuum of special theory of relativity and prove that it is Lorentz invariant.

Solution : If $p(p_x, p_y, p_z)$ be the momentum and E be the energy of space-time continuum then the energy momentum vector in it of special theory ofrelativity is defined by $\frac{E^2}{c^2} - p^2 \left(\text{or } p^2 - \frac{E^2}{c^2}\right)$.

$\therefore$ We are to prove that this energy momentum vector is Lorentz invariant.

Let S and S′ be two inertial frames of reference where S′ moves with constant speed 'v' in the direction of x-axis relative to S, the two origins are coincidental at t = t′ = 0.

$\therefore$ If $p(p_x, p_y, p_z)$ and $p'(p_x', p_y', p_z')$ be the momentums w.r.t. S and S′ then

$$p_x' = \beta\left(p_x - \frac{Ev}{c^2}\right), \qquad p_y' = p_y, \qquad p_z' = p_z$$

and $E' = \beta(E - vp_x)$ where E is the energy and

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \quad \frac{E'^2}{c^2} - p'^2 = \frac{1}{c^2}\beta^2(E - vp_x)^2 - \left(p_x'^2 + p_y'^2 + p_z'^2\right)$$

$$= \frac{\beta^2}{c^2}(E - vp_x)^2 - \beta^2\left(p_x - \frac{vE}{c^2}\right)^2 - p_y^2 - p_z^2$$

$$= \beta^2\left[\frac{1}{c^2}\left(E^2 - 2vEp_x + v^2p_x^2\right) - \left(p_x^2 - \frac{2v}{c^2}Ep_x + \frac{v^2E^2}{c^4}\right)\right] - p_y^2 - p_z^2$$

$$= \beta^2\frac{E^2}{c^2}\left(1 - \frac{v^2}{c^2}\right) - \beta^2p_x^2\left(1 - \frac{v^2}{c^2}\right) - p_y^2 - p_z^2 \qquad \left(\because \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

$$= \frac{E^2}{c^2} - \left(p_x^2 + p_y^2 + p_z^2\right)$$

$$\therefore \frac{E'^2}{c^2} - p'^2 = \frac{E^2}{c^2} - p^2$$

Hence the energy momention vector thus define is Lorentz invariant.

3.7. A body of mass M while at rest disintegrates into two masses M_1 and M_2. Show that the energy of the two parts E_1 and E_2 are $\frac{M^2}{2c^2}\left(M^2+M_1^2-M_2^2\right)$, $\frac{M^2}{2c^2}\left(M^2+M_2^2-M_1^2\right)$

Sol. It E is the total energy of the body then

$$E = E_1 + E_2 = MC^2 \tag{1}$$

But $M_1 + M_2 = M$ (2)

Again from energy momentum vector of special theory of relativity,

we have $\frac{E^2}{c^2} - p^2$ = constant (i.e. invariant)

or, $\frac{\left(m_0c^2\right)^2}{c^2} - 0$ = constant $[E = m_0c^2 + 0]$

So, Constant $= m_0c^2$

$$\therefore \frac{E^2}{c^2} - p^2 = m_0c^2$$

$\therefore$ For the two parts of rest masses M_1 and M_2 we have

$$\frac{E_1^2}{c^2} - p^2 = M_1^2c^2$$

$$\frac{E_2^2}{c^2} - p^2 = M_2^2c^2$$

Substracting

$$\frac{1}{c^2}\left(E_1^2 - E_2^2\right) = \left(M_1^2 - M_2^2\right)c^2$$

$$or \quad \frac{1}{c^2}\left(E_1 + E_2\right)\left(E_1 - E_2\right) = c^2\left(M_1^2 - M_2^2\right)$$

$$or \quad E\left(E_1 - E_2\right) = c^4\left(M_1^2 - M_2^2\right) \qquad [\because E = E_1 + E_2]$$

$$\therefore \quad E_1 - E_2 = \frac{c^4}{E}\left(M_1^2 - M_2^2\right) = \frac{c^2}{M}\left(M_1^2 - M_2^2\right) \qquad \left(\because E = Mc^2\right)$$

But $E_1 + E_2 = Mc^2\ (= E)$

Adding, $E_1 = \frac{c^2}{2M}\left(M_1^2 - M_2^2 + M^2\right)$

Substracting, $E_2 = \frac{c^2}{2M}\left(M_2^2 - M_1^2 + M^2\right)$

3.8 Relativistic Lagrangian and Hamiltonian :-

Relativistic Lagrangian : In classical mechanies the Lagrangian is defined as

$$L = T - V \qquad (1)$$

where T is K.E of the system and is the function of generalized momentum P_r or of generalized velocities $\dot{q}_r$; while potential V is the function of generalized co-ordinates q_r only. We also know from classical mechanics, that

$$P_r' = \frac{\partial L}{\partial q_r} = \frac{\partial (T - v)}{\partial \dot{q}_r} = \frac{\partial T}{\partial \dot{q}_r} \qquad (2)$$

Since $\frac{\partial v}{\partial \dot{q}_r} = 0$, because P.E. does not depend upon the generalized velocity

$$\therefore \frac{dp_r}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) \qquad (3)$$

But according to Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$$

$\therefore$ Equation (3) accomes

$$\frac{dp_r}{dt} = \frac{\partial L}{\partial q_r} \qquad (4)$$

We have seen that the relativistic K.E of a particle of rest mass m_o and moving with a velocity v is given by

$$T = (m - m_0)c^2 = m_0 c^2 \left[\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right] \qquad (5)$$

and relativistic momentum

$$P = mv \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6}$$

By definition

$$\frac{\partial T}{\partial \dot{x}} = p_x, \qquad \frac{\partial T}{\partial \dot{y}} = p_y, \qquad \frac{\partial T}{\partial \dot{z}} = p_z \tag{7}$$

We see that by actual differentiation of equation (5) it is found that

$$\frac{\partial T}{\partial \dot{x}}, \quad \frac{\partial T}{\partial \dot{y}} \quad \text{and} \quad \frac{\partial T}{\partial \dot{z}}$$

donot give the respective relativistic momentum components i.e.

$$\left.\begin{aligned} P_x &= \frac{m_0 \dot{x}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\partial T^*}{\partial \dot{x}} \\ P_y &= \frac{m_0 \dot{y}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\partial T^*}{\partial \dot{y}} \\ P_z &= \frac{m_0 \dot{z}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\partial T^*}{\partial \dot{z}} \end{aligned}\right] \tag{8}$$

Here T* denotes a function such that when differentiated w.r to x, y or z will give P_x, P_y or P_z respectively.

As K.E. T* is a function of velocity components $\dot{x}$, $\dot{y}$ or $\dot{z}$, i.e.,

$$T^* = T^*(\dot{x}, \dot{y}, \dot{z})$$

$$\therefore \quad dT^* = \frac{\partial T^*}{\partial \dot{x}} d\dot{x} + \frac{\partial T^*}{\partial \dot{y}} d\dot{y} + \frac{\partial T^*}{\partial \dot{z}} d\dot{z}$$

$$= \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \left(\dot{x}d\dot{x} + \dot{y}d\dot{y} + \dot{z}d\dot{z}\right) \tag{9} \quad \text{[by (8)]}$$

But $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

Differentiating we have $2v\,dv = 2\dot{x}d\dot{x} + 2\dot{y}d\dot{y} + 2\dot{z}d\dot{z} \Rightarrow v\,dv = \dot{x}d\dot{x} + \dot{y}d\dot{y} + \dot{z}d\dot{z}$

this in equation (9) we get

$$dT^* = \frac{m_0 v\,dv}{\sqrt{1-\frac{v^2}{c^2}}} \quad (10)$$

Substituting $1-\frac{v^2}{c^2} = t$

Differentiating we have $-\frac{2v\,dv}{c^2} = dt \qquad \therefore v\,dv = -\frac{c^2}{2}dt$

Equation (10) gives

$$dT^* = -\frac{m_0c^2}{2}\frac{dt}{t^{\frac{1}{2}}}$$

Integrating we get

$$T^* = -m_0c^2 t^{\frac{2}{2}} + A, \qquad \text{where A is a constant of integration}$$

$$= -m_0c^2\sqrt{1-\frac{v^2}{c^2}} + A \quad (11)$$

since $t = 1-\frac{v^2}{c^2}$

We know that when

$$v << c, \qquad T^* = \frac{1}{2}m_0v^2$$

Equation (11) gives

$$\frac{1}{2}m_0v^2 = -m_0c^2 + A \quad \text{or} \quad A = \frac{1}{2}m_0v^2 + m_0c^2 = m_0c^2 \qquad [\because v << c]$$

Substituting the values of A in equation (11), we get

$$T^* = -m_0c^2\sqrt{1-\frac{v^2}{c^2}} + m_0c^2$$

$$= m_0c^2 \left\{1 - \sqrt{1 - \frac{v^2}{c^2}}\right\} \tag{12}$$

∴ The relativistic Lagrangian function

$$L = T^* - V, \qquad \text{V being potential energy}$$

and $T = T^*$ in this case

$$\therefore \; L = m_0c^2 \left\{1 - \sqrt{1 - \frac{v^2}{c^2}}\right\} - V \tag{13}$$

Now we can write the Lagrangian form of relativistic equation of motion i.e., analogue to classical equation (4)

From equation (12) and (13), we see that

$$\frac{\partial L}{\partial \dot{q}_r} = \frac{\partial T^*}{\partial \dot{q}_r}$$

$$\text{and} \quad \frac{\partial L}{\partial q_r} = -\frac{\partial V}{\partial q_r} \tag{14}$$

∴ The relativistic form of equation (4) is

$$\frac{\partial P_r}{\partial t} = -\frac{\partial V}{\partial q_r} \qquad \text{using (14)}$$

or, in terms of components

$$\left.\begin{aligned}
&\frac{d}{dt}\left(\frac{m_0\dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = -\frac{\partial V}{\partial x} \\
&\frac{d}{dt}\left(\frac{m_0\dot{y}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = -\frac{\partial V}{\partial y} \\
\text{and} \quad &\frac{d}{dt}\left(\frac{m_0\dot{z}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = -\frac{\partial V}{\partial z}
\end{aligned}\right\} \tag{16}$$

Relativistic Hamiltonian :

The relativistic Hamiltonian function H is defined as

$$H = \Sigma P_x \dot{x} - L$$

$$= \Sigma \frac{\partial T^*}{\partial \dot{x}} \dot{x} - L = \Sigma \frac{\partial T^*}{\partial \dot{x}} \dot{x} - m_0 c^2 \left[1 - \sqrt{1 - \frac{v^2}{c^2}}\right] + V$$

Substituting value of T* from (12), we get

$$H = \Sigma\left[\frac{\partial}{\partial \dot{x}}\left\{m_0 c^2\left(1 - \sqrt{\frac{v^2}{c^2}}\right) - V\right\}\right]\dot{x} \ -m_0 c^2\left\{1 - \sqrt{1 - \frac{v^2}{c^2}}\right\} + V$$

$$= m_0 c^2 \Sigma \frac{\partial}{\partial \dot{x}}\left\{\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}}\right\}\dot{x} - m_0 c^2\left\{1 - \sqrt{1 - \frac{v^2}{c^2}}\right\} + V$$

$$= \Sigma \frac{m_0 \dot{x}^2}{1 - \frac{v^2}{c^2}} - m_0 c^2\left\{1 - \sqrt{1 - \frac{v^2}{c^2}}\right\} + V$$

$$= \frac{m_0\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2\left\{1 - \sqrt{1 - \frac{v^2}{c^2}}\right\} + V$$

$$= \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2\left\{1 - \sqrt{1 - \frac{v^2}{c^2}}\right\} + V$$

$$= m_0 c^2\left[\frac{\frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 + \sqrt{1 - \frac{v^2}{c^2}}\right] + V$$

$$= m_0 c^2\left[\frac{\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right] + V.$$

$$= m_0c^2\left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right]+V \qquad (17)$$

This is Hamiltonian relativistic function.

Clearly H = T* + V from (12)

= Relativistic K.E + Potential Energy.

Thus like classical Hamiltonian, relativistic Hamiltonian also denotes the total energy of the system.

We have

$$P_x = \frac{m_0\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad P_y = \frac{m_0\dot{y}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{and} \qquad P_z = \frac{m_0\dot{z}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \quad P_x^2 + P_y^2 + P_z^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}}\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]$$

$$= \frac{m_0^2v^2}{1-\frac{v^2}{c^2}} = -\,m_0^2c^2 + \frac{m_0^2c^2}{1-\frac{v^2}{c^2}}$$

$$\text{or} \qquad P_x^2 + P_y^2 + P_z^2 + m_0c^2 = \frac{m_0^2c^2}{1-\frac{v^2}{c^2}}$$

$$\text{or} \qquad \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{\frac{P_x^2 + P_y^2 + P_z^2 + m_0^2c^2}{m_0^2c^2}} \qquad (18)$$

Substituting this value of $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ in equation (17), the expression for the relativistic Hamiltonian becomes

$$H = m_0c^2\left[\sqrt{\left\{\frac{P_x^2 + P_y^2 + P_z^2 + m_0^2c^2}{m_0^2c^2}\right\}}-1\right]+V$$

or, $$H=\left[c\sqrt{\{P_x^2+P_y^2+P_z^2+m_0^2c^2\}}-m_0c^2+V\right] \quad (19)$$

This is the required expression for the relativistic Hamiltonian.

3.9 Particle with Zero mass :

Let m be the mass, u be the velocity and E be the energy and p be the momentum of a particle in a coordinate system S. Then

$$\frac{E^2}{c^2}-p^2=m_0^2c^2 \quad \text{where } m_0 \text{ is the rest mass.}$$

$$\text{If } m_0=0 \text{ then } \frac{E^2}{c^2}-p^2=0 \quad \Rightarrow p=\frac{E}{c} \quad (1)$$

$$\text{Again } p = mu = \frac{E}{c^2}u \qquad (\because E=mc^2) \quad (2)$$

Now equation (1) & (2)

$$\frac{E}{c}=\frac{E}{c^2}u$$

$\therefore$ If rest mass is zero, the particle will move with a velocity equal to the velocity of light.

Also its momentum and mass are given by $p=\frac{E}{c}$ and $m=\frac{E}{c^2}$.

Ex. Show that $\frac{E^2}{c^2}-p^2=m_0^2c^2$

Sol. $\frac{E^2}{c^2}-p^2=\frac{(mc^2)^2}{c^2}-(p_x^2+p_y^2+p_z^2)=m^2c^2-[(mu_x)^2+(mu_y)^2+(mu_z)^2]$

$$=m^2c^2-m^2(u_x^2+u_y^2+u_z^2)=m^2c^2-m^2u^2$$

$$=\left[\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\right]^2.c^2\left(1-\frac{u^2}{c^2}\right)$$

$$=\frac{m_0^2}{\left(1-\frac{u^2}{c^2}\right)}\times c^2\left(1-\frac{u^2}{c^2}\right)=m^2c^2$$

Ex. Show that $E^2 - p^2c^2$ is Lorentz invariant.

Sol. We have,

$$\frac{E^2}{c^2} - p^2 = m_0^2c^2$$

$$\Rightarrow E^2 = p^2c^2 + m_0^2c^4 \qquad (1)$$

The relation (1) is Lorentz invariant because we have proved earlier

$$p'^2 - \frac{E'^2}{c^2} = p^2 - \frac{E^2}{c^2}$$

$$\Rightarrow p^2 - \frac{E^2}{c^2} \text{ is Lorentz invariant}$$

$$\Rightarrow c^2p^2 - E^2 \text{ is Lorentz invariant}$$

But m_0 and c are also invariant

$$\Rightarrow E^2 - c^2p^2 = m_0^2c^4 \text{ is Lorentz invariant.}$$

●●●

Unit 4

4.1 Introduction

In relatiristics mechanics space time continuum is to be characterized by four numbers (x_1, x_2, x_3, x_4). The set of values is said to be constitute the Minkowski's world of four dimensions. Minkowski defined the corresponding line element as

$$dx^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 \quad (1)$$

which is to generate the Minkowskis World of 4 dimensions. But in space time continuum the accepted metric is

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2 (dt)^2 \quad (2)$$

Comparing (1) and (2)

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = ict$$

This shows that the time co-ordinate is imaginary which can not be measurud in the same scale of the space co-ordinate. This space is termed as space time continuum.

4.2 Geometrical reprenentation of simultaneity, contraction and dilation.

Now we are in a position to explain geometrially more relativistic phenomena yiz. (1) relativity of simultaneity, (2) length contraction and (3) time dilation.

1 Relativity of simultaneity :

Two events, p_1 and p_2, occure simultaneously i.e. at the same value of ct, [Fig 4.2(a)]. But the figure shows that they occur according to the s′- system, at two different values of ct′. This explains that simultaneity is a relative concept. In the present instance, there is simultancity in the S-system, but not so according to the S′-system.

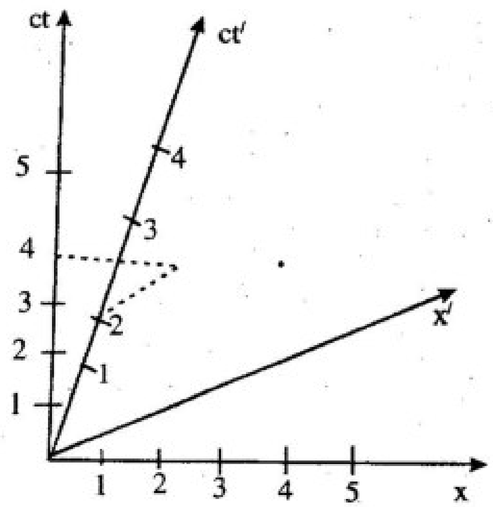


Fig. 4.2(a)

2 Length Contraction

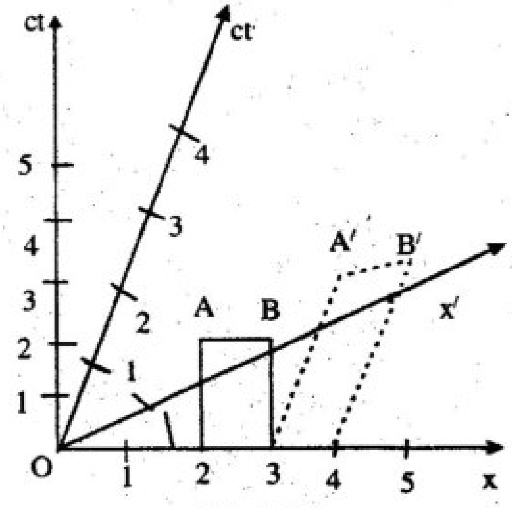


Fig. 4.2(b)

Fig. 4.2 (b) shows that AB is a rod of unit length in the S-system. But it appears to be shorter length about 0.8 according to S'. In otherwords, the lengths of the rod appear to be shorter according to the S'-system. Correspondingly, a length A'B' in the S'-system appear to be shorter according to the S-system. This is the phenomemenon of length contraction.

3 Time Dilation

Figure 4.2(c) shows that there is unit (ct) interval between two events. C and D, in the S-system. According to the S' system the (ct') interval appears to be longer. Correspondingly, a time interval between two events, C' and D', in the S'-system appear to be longer according to the S-system. This is the phenomenon of time dilation.

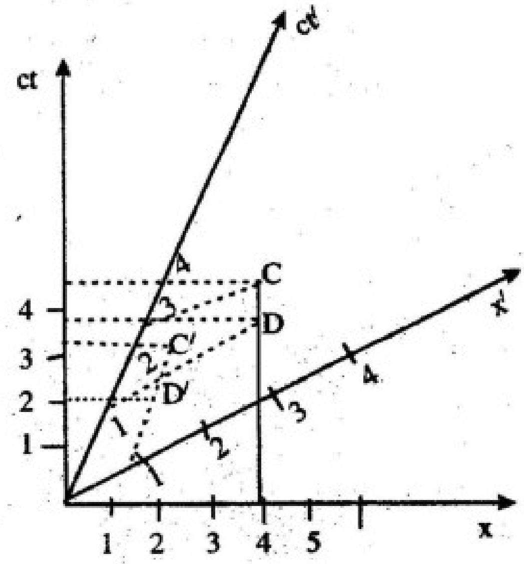


Fig. 4.2(c)

4.3 Space like and time like intervals–

In space time continuum an event is specified by (x, y, z, ict) customerily by (x, y, z, t). Therefore an interval between two eyents characterised by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) is defined as

$$S_{12}^2 = c^2(t_2 - t_1)^2 - [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \qquad (1)$$

in an inertial frame S.

In a frame S' (with usual meaning)

$$S_{12}'^2 = c^2(t_2' - t_1')^2 - [(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2]$$

From Lorentz Transformation, we get,

$$x_1' = \beta(x_1 - vt_1), \quad y_1' = y_1, \quad z_1' = z_1, \quad t_1' = \beta\left(t_1 - \frac{vx_2}{c^2}\right)$$

$$x_2' = \beta(x_2 - vt_2), \quad y_2' = y_2, \quad z_2' = z_2, \quad t_2' = \beta\left(t_2 - \frac{vx_2}{c^2}\right)$$

$$\therefore S_{12}'^2 = c^2\left[\beta^2\left\{(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right\}^2\right] - \left[\beta^2\{(x_2 - x_1) - v(t_2 - t_1)\}^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]$$

$$= c^2\beta^2\left[(t_2 - t_1)^2 + \frac{v^2}{c^4}(x_2 - x_1)^2 - \frac{2v}{c^2}(t_2 - t_1)(x_2 - x_1)\right]$$

$$-\beta^2\left[(x_2 - x_1)^2 + v(t_2 - t_1)^2 - 2v(x_2 - x_1)(t_2 - t_1)\right] - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

$$= c^2\beta^2(t_2 - t_1)^2\left(1 - \frac{v^2}{c^2}\right) - \beta^2(x_2 - x_1)^2\left(1 - \frac{v^2}{c^2}\right) - (y_2 - y_1)^2$$

$$= c^2(t_2 - t_1)^2 - \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]$$

$$\therefore S_{12}'^2 = S_{12}^2 \Rightarrow S_{12}' = S_{12}$$

This shows that the interval defined by (1) remains Lorentz invariant.

Time like interval :

If $S_{12}^2 > 0$

i.e. $c^2(t_2 - t_1)^2 > (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ (1)

in that case the interval is real.

This shows that if a light signal is emitted from the site of the first event moment it occurs reaches the site of the 2nd event before it occurs.

Let us search for events in the moving frame S' in which (1) holds good. Let us consider two events in S'-frame which occur at the same position but at different times, so that

$t_2' \neq t_1'$ but $x_2' = x_1'$, $y_2' = y_1'$, $z_2' = z_1'$.

$$\therefore \quad S_{12}'^2 = c^2(t_2' - t_1') - 0 - 0 - 0 \quad (2)$$

$$\text{But } t_2' = \beta\left(t_2 - \frac{vx_2}{c^2}\right), \quad t_1' = \beta\left(t_1 - \frac{vx_1}{c^2}\right)$$

So, $$(t_2' - t_1')^2 = \beta^2\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]^2 > 0$$

Since $S_{12}'^2 > 0$ i.e. $(t_2' - t_1')^2 > 0.$

$$\Rightarrow (t_2 - t_1) > \frac{v}{c^2}(x_2 - x_1)$$

$$\Rightarrow c(t_2 - t_1) > \frac{v}{c}(x_2 - x_1)$$

$$\because v << c \quad \text{i.e.} \quad \frac{v}{c} << 1$$

$$\therefore c^2(t_. - t_1)^2 > (x_2 - x_1)^2$$

which is applicable since due to Lorentz transformation $y_2 = y_1$ & $z_2 = z_1$.

∴ The condition of real interval i.e. $S_{12}^2 > 0$ is satisfied by those events in S'-frame which occur at the same position but at different times. Again this interval contains only time component and so it is called time like interval, otherwise real interval is called time like interval.

4.4 Position, Four vectors, Four velocity, Four Forces and Four momentums.

The four-dimensional space of Minkowski is the set of points (x_1, x_2, x_3, x_4) and the corresponding metric is taken as

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \qquad (1)$$

But in space time continuum are know that

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \qquad (2)$$

which remains invariant subject to Lorentz transformation.

∴ To fit the metric as adoptable in space time continuum,we an get

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = ict$$

Hence (2) is taken as the appropriate metric in space time continuum of special theory of relativity.

∴ (x_1, x_2, x_3, x_4) i.e. (x, y, z, ict) or (x, y, z, t) (customarily) is called the position four vector in space time continuum.

If (u_1, u_2, u_3, u_4) are the corresponding components of velocity in the space time continuum, then it is called the four-velocity.

Here $u_1 = \frac{dx_1}{dT}$, $u_2 = \frac{dx_2}{dT}$, $u_3 = \frac{dx_3}{dT}$, $u_4 = \frac{dx_4}{dT}$ being the local time given by

$$dT = \sqrt{1-\frac{u^2}{c^2}}dt \qquad \left[\text{from } \Delta t' = \beta\Delta t \text{ and } \sqrt{1-\frac{u^2}{c^2}}\,\Delta t = \Delta T\right]$$

$$u_1 = \frac{dx}{dt}.\frac{dt}{dT} = \frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$u_2 = \frac{dx_2}{dT} = \frac{dy}{dt}.\frac{dt}{dT} = \frac{u_y}{\sqrt{1-\frac{u^2}{c^2}}}$$

Similarly

$$u_3 = \frac{u_z}{\sqrt{1-\frac{u^2}{c^2}}}, u_4 = \frac{dx_4}{dT} = \frac{d}{dt}(ict) \times \frac{dt}{dT} = \frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}.$$

This shows that $u_x = u_y = u_z = 0$, $u_4 \neq 0$; $u_1 = u_2 = u_3 = 0$

This shows that if a particle or body is at rest in Newtonian sense of three dimension the corresponding four velocity is non zero.

Geometrically if a body is at rest in three dimensional space but it is not at rest in four dimensional space. This is called the geometrical meaning.

4.5 Relativistic equations of motion.

In Newtonian mechanies, the equation of motion of a particle of linear momentum p and rest mass m_0 is given by

$$F = \frac{dp}{dt} = \dot{p} = \frac{d}{dt}(m_0 v) \qquad \text{(i)}$$

As time t of particle vectors from one fnertial frame to another. Therefore this expression is not Lorentz invariant. Here we try to generalise equation (1) so that it may satisfy the principle of velocity. In Newtonian mechanics time t to plays the role of invariant parameter, but in relativistic time t is variable and similar parameter is length ds, which is given by

$$ds^2 = c^2dt^2 - dx^2 - dx^2 - dy^2 - dz^2$$

This implies

$$\left(\frac{ds}{dt}\right)^2 = c^2 - \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}$$

$$= c^2 - u^2$$

where u is ordinary velocity of particle

$$u^2 = \sqrt{u_x{}^2 + u_y^2 + u_z^2}$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{c^2 - u^2} = c\sqrt{1 - \frac{u^2}{c^2}} \qquad (2a)$$

$$\Rightarrow \frac{ds}{c} = dt\sqrt{1 - \frac{u^2}{c^2}} \quad \text{or} \quad \frac{ds}{c} = \frac{dt}{\gamma} \qquad (2b)$$

$$\Rightarrow ds = \frac{cdt}{\gamma} \qquad (3)$$

Physical significance of ds is that apart from the factor c, it is proper time of the partide in a frame in which it is intantaneously of rest.

The momentum components in cartesian co-ordinate may be expressed as

$$p_x = mu_x = \frac{m_0 u_x}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma\, m_0 u_x$$

Similarly $\quad p_y = m_0 c.\dfrac{dy}{ds}$

$$p_z = m_0 c\frac{dz}{ds}$$

In four dimensional space, the result may be generalized as four momentum

$$p^\mu = m_0 c\frac{dx^\mu}{ds} \qquad (4)$$

Therefore in analogy with Newtonian mechanics, the equation of motion is relatively would become

$$p^\mu = k^\mu \qquad (5)$$

where k^μ is four force or Minkowski force and dot means differentiation w.r to invariant parameter 's'

$$\Rightarrow \frac{d}{ds}\left(m_0 c \frac{dx^\mu}{dx}\right) = K^\mu \tag{6}$$

This equation is called relativistic equation of motion or Minkowski equation of motion.
Classical limit of Minkowski equation of motion–

In classical limit $\frac{v}{c} << 1$

As for any function f, $\frac{df}{ds} = \frac{df}{dt} \cdot \frac{dt}{ds}$

or, $$\frac{d}{dt}\left(m_0 c \frac{dx^u}{ds}\right) = K^u \frac{ds}{dt} \tag{7}$$

Again writting

$\frac{dx^u}{ds} = \frac{dx^u}{dt} \cdot \frac{dt}{ds}$, equation (7) takes the form

$$\frac{d}{dt}\left(m_0 c \frac{dx^u}{dt} \cdot \frac{dt}{ds}\right) = K^u \frac{ds}{dt} \tag{8}$$

The space part of equation (8) may be expressed as

$$\frac{d}{dt}\left(m_0 c \frac{dt}{ds} \frac{dx_i}{dt}\right) = K_i \frac{ds}{dt} \quad \text{where} \quad i = 1,\ 2,\ 3. \tag{9}$$

Comparing this with Newtonian equation of motion

$$\frac{d}{dt}\left(m \frac{dx_i}{dt}\right) = F_i \tag{10}$$

We note that

$$m = m_0 c \frac{dt}{ds} = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad \text{u sin g(2a)} \tag{11}$$

and $K_i \frac{ds}{dt} = F_i$ i.e. $K \frac{ds}{dt} = F_i$ (12)

where K is spatial part of four vectors K^μ.

Further in the limit

$$\frac{v}{c} \to 0, \qquad \frac{ds}{dt} = c$$

$$\text{and} \quad K = \frac{F}{c} \tag{13}$$

i.e. K and F are essentially the same. Equation (11) implies that mass varies with velocity of particle, but there is no internal mechanism is increase the mass of the particles however it is due to equivalence of mass and energy. Accordingly the increase in mass is due to increased kinetic energy of particle. This suggests that we can correct the classical equation of motion by using a variable mass m in place of rest mass m_0. But the Minkowski form of equation is the fundamental relativistic formulation of equation of motion.

4.6 Covariant four dimensional formulation of the laws of mechanies.

If the form of a law is not changed by a certain co-ordinate transformatioon, the law is said to be invariant or covariant. If any physical law may be expressed in a covariant four dimensional form, the law will be invariant under Lorentz transformations. The covariant four dimensional formulation of the laws of mechanics has great significance in the development of the theory of relativity. Each law of nature expreses a certain relation between physical quantities. Let A, B, ... be a set of such physical quantities measured in inertial systems. Let some quantities among A, B,.... depend upon the space time co-ordinate x_i in systems. Thus a certain law of nature may be expressed by one or several equations of the form.

$$f\left(A, B \ldots\ldots \frac{\partial A}{\partial x_i}, \frac{\partial B}{\partial x_j}\right) = 0 \tag{1}$$

Where f is a function of the quantities A, B and possible of their derivatives w.r. to the space time co-ordinates.

Let the above mentioned quantities have values A′, B′ ..... when measured in inertial system S′. Then the physical law expressed by (1) in system S can be represented in system S′ by the equation of the form

$$f\left(A', B' \ldots\ldots \frac{\partial A}{\partial x_i'}, \frac{\partial B}{\partial x_j'}\right) = 0 \tag{2}$$

where x_i' are the space time co-ordinate in system S′.

According to special theory of relativity the function f in (2) must be the same function of physical quantities (A', B'.....) as is the function f of (A, B,...) in (1). That is any relation between physical quantities must be expressed by means of form invarient or covariant equations.

As Lorentz transformations represents rotations in (3 + 1) dimensional space, it is natural to attempt to meet the requirements of covariance of the law of nature under these transformations by a generalization of three dimensional vectors and tensors of four dimensions and to write the fundamental equations in four dimensional form.

As an example consider the position vector of a point in four dimensions which is expressed as

$$\bar{r} = xi + yj + zk + p\hat{p}$$

Taking its dot product with itself we shall get a world scalar and hence Lorentz invariant i.e.

$$\bar{r} . \bar{r} = x^2 + y^2 + z^2 + p^2$$

but $p^2 = -c^2 t^2$

$$\therefore \ \bar{r}^2 = x^2 + y^2 + z^2 - c^2t^2 = \text{Lorentz invariant.}$$

As another example, consider scalar wave equation of the type

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

Like three dimensional gradient, we have a four dimensional operator called D' Alembertian with components.

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial p} \qquad \text{i.e.} \quad \Box = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} + \hat{p}\frac{\partial}{\partial p}$$

Taking its scalar product with itself weget a scalar quantity and hence Lorentz invariant, i.e.

$$\Box \cdot \Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial p^2} = \text{Lorentz invariant}$$

$$\text{or,} \quad \Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad [\because p = ict]$$

$$= \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \text{Lorentz invariant}$$

Thus $\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}$ is invariant under Lorentz transformations.

From above example it is clear that "Any physical law will be invariant under Lorentz transformations it may be expressed in a covariant four dimensional form."

●●●

Unit : 5

5.1 Introduction

Electrodynamics is that branch of physics in which the force acting between two moving charges depends on their masses and valocities.

The force acting between two charges q_1 and q_2 is given by the Columb's law $F = \gamma \frac{q_1 q_2}{r^2}$

5.2 Intensity :

The force per unit charge 'q' arou'.d a charged body Q of an electric field is called the intensity E of the field.

$$E = \frac{F}{Q}$$

5.3 Current density (Flux of fluid) :

The amount of charges passing across a unit area of cross-section moving with velocity 'V' per unit of time is called curreat density.

If 'n' is the number densiity and q is the charge, then

$$\vec{\tau} = \text{total charge} \times \text{volume}$$

$$= (nq)\,\vec{v} = \sigma\vec{v} \quad \text{where } \sigma = nq \text{ is called the space charge.}$$

5.4 Maxwell's equation :

Orested has first observed that if there is a stationary electric field $\overline{E}$ and if an observer S' moves with constant speed 'v' w.r.t. first observer in S, then the observer in S' will experience, the presence of both electric and magnetic field whereas the observer in S experiences the presence of the stationary electric field only. The electric field $\overline{E}$ and magnetic field $\overline{B}$ (or $\overline{H}$) are connected by a set of equations which are formulated as follows by Maxwell.

In esu unit :

$$\nabla.\overline{E} = 4\pi\sigma \qquad \text{(A)}$$

$$\nabla.\overline{B} = 0 \qquad \text{(B)}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{B} = \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t} + \frac{4\pi}{c^2}\bar{J}$$

In Gaussian units, they are

$$\nabla.\bar{E} = 4\pi\sigma \qquad (1)$$

$$\nabla.\bar{H} = 0 \qquad (2)$$

$$\nabla \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t} \qquad (3)$$

$$\nabla \times \bar{H} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t} + \frac{4\pi}{c}\bar{J} \qquad (4)$$

This set of equations is called the Maxwell's equation for electro magnetic field.

Taking divergence of (3)

$$\nabla.\left(\nabla \times \bar{E}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \times \bar{H}\right)$$

$$0 = -\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla.\bar{H}\right)$$

which is equivalent to $\nabla.\bar{H} = 0$ $\left(\nabla.\bar{H} = \bar{C}\right)$

Again taking divergence of (4)

$$\nabla.\left(\nabla \times \bar{H}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \times \bar{E}\right) + \frac{4\pi}{c}\left(\nabla.\bar{J}\right)$$

$$\Rightarrow 0 = \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla.\bar{E}\right) + \frac{4\pi}{c}\left(-\frac{\partial J}{\partial t}\right)$$

$$\therefore \quad \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla.\bar{E} - 4\pi\sigma\right) = 0$$

which is equivalent to $\nabla.\bar{E} - 4\pi\sigma$.

This shows that the equations (1) and (2) are recoverable from equation (3) and (4). This amounts to saying that all the four Maxwell's equations are not independent. Otherwise there are the independent equations in the set of Maxwell's equations.

5.5 Q. Transformation relations of differential operators in correction to Lorentz transformation.

Sol. From Lorentz transformation equations we have

$$x' = \beta(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \beta\left(t - \frac{vx}{c^2}\right)$$

or, $$x = \beta(x' + vt'), \qquad y' = y, \qquad z' = z, \qquad t = \beta\left(t' + \frac{vx'}{c^2}\right) \qquad (1)$$

$$\because \quad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x'}$$

$$= \beta\frac{\partial}{\partial x} + 0 + 0 + \frac{\beta\upsilon}{c^2}\frac{\partial}{\partial t}$$

$$= \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right) \qquad \text{[from (1)]}$$

Also,

$$\frac{\partial}{\partial y'} = 0 + \frac{\partial}{\partial y}.1 + 0 + 0 = \frac{\partial}{\partial y}$$

Similarly $\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$.

Again,

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial t'}$$

$$= \beta v\frac{\partial}{\partial x} + 0 + 0 + \beta\frac{\partial}{\partial t} = \beta\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)$$

Hence,

$$\frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right), \qquad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial t'} = \beta\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)$$

which are the connection (transformation) of 1st order differential operator w.r.t. Lorentz transformation.

5.6 D' Alembert's operator :

$$\Box^2\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$=\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi$$

where $\Box \equiv \nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called the D' Alemberts operator.

5.7 Transformation of $\overline{E}$ (E_x, E_y, E_z) and $\overline{H}$ (H_x, H_y, H_z)

Let S and S′ be two inertial frames where S′ moves with constant speed υ in the direction of x-axis relative to S; the two origins coincide at $t = t' = 0$.

Also, we know that the Maxwell's equations are satisfied by

$$\overline{H} = \text{Curl}\,\overline{A} \qquad (1)$$

$$\& \qquad \text{grad}\,\phi = -\overline{E}-\frac{1}{c}\frac{\partial\overline{A}}{\partial t} \qquad (2)$$

where $\overline{A}$ is the vector potential and ϕ is the scalar potential.

∴ From (1)

$$(iH_x + jH_y + kH_z) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

so that

$$H_x = \frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}, \quad H_y = \frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}, \quad H_z = \frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y} \qquad (3)$$

Also from (2)

$$E_x = -\frac{\partial\phi}{\partial x}-\frac{1}{c}\frac{\partial A_x}{\partial t} \qquad (4)$$

with other two similarities.

But the transformation relations of $\overline{A}$ (A_x, A_y, A_z) and ϕ are

$$A'_x = \beta\left(A_x-\frac{v}{c}\phi\right), \quad A'_y = A_y, \quad A'_z = A_z$$

$$\phi' = \beta\left(\phi - \frac{v}{c}A_x\right) \tag{5}$$

and

$$\frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right), \quad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t'} = \beta\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) \tag{6}$$

Now

$$H'_x = \frac{\partial A'_z}{\partial y'} - \frac{\partial A'_y}{\partial z'} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = H_x$$

$$H'_y = \frac{\partial A'_x}{\partial z'} - \frac{\partial A'_z}{\partial x'}$$

$$= \beta\frac{\partial}{\partial z}\left(A_x - \frac{v}{c}\phi\right) - \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)A_z$$

$$= \beta\left[\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{v}{c}\left(-\frac{\partial \phi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t}\right)\right]$$

$$= \beta\left[H_y + \frac{v}{c}E_y\right]$$

Similarly,

$$H'_z = \beta\left(H_z - \frac{v}{c}E_y\right)$$

Now

$$E'_x = -\frac{\partial \phi'}{\partial x'} - \frac{1}{c}\frac{\partial A'_x}{\partial t'}$$

$$= -\beta^2\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(\phi - \frac{v}{c}A_x\right)$$

$$= -\frac{1}{c}\beta^2\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(A_x - \frac{v}{c}\phi\right)$$

$$= -\beta^2\left[\left(\frac{\partial \phi}{\partial x} - \frac{v}{c}\frac{\partial A_x}{\partial x} + \frac{v}{c^2}\frac{\partial \phi}{\partial t} - \frac{v^2}{c^3}\frac{\partial A_x}{\partial t}\right) + \frac{1}{c}\left(\frac{\partial A_x}{\partial t} - \frac{v}{c}\frac{\partial \phi}{\partial t} + v\frac{\partial A_x}{\partial x} - \frac{v^2}{c}\frac{\partial \phi}{\partial x}\right)\right]$$

$$= -\beta^2\left[\frac{\partial \phi}{\partial x}\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c}\frac{\partial A_x}{\partial t}\left(1 - \frac{v^2}{c^2}\right)\right]$$

$$E'_x = -\frac{\partial \phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} \qquad \because \beta^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

$$\therefore\ E'_x = E_x$$

Again $$E'_y = -\frac{\partial \phi'}{\partial y'} - \frac{1}{c}\frac{\partial A'_y}{\partial t'}$$

$$= -\beta\frac{\partial}{\partial y}\left(\phi - \frac{v}{c}A_x\right) - \beta\frac{1}{c}\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)A_y \qquad \text{[using (5) and (6)]}$$

$$= \beta\left[\left(-\frac{\partial \phi}{\partial y} - \frac{1}{c}\frac{\partial A_y}{\partial t}\right) + \frac{v}{c}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)\right]$$

$$= \beta\left(E_y - \frac{v}{c}H_z\right)$$

Similarly, $$E'_z = \beta\left(E_z + \frac{v}{c}H_y\right)$$

Thus the transformation relations of $\overline{E}(E_x, E_y, E_z)$ and $\overline{H}(H_x, H_y, H_z)$ are

$$E'_x = E_x \qquad H'_x = H_x$$

$$E'_y = \left(E_y - \frac{v}{c}H_z\right)\beta \qquad H'_y = \beta\left(H_y + \frac{v}{c}E_z\right)$$

$$E'_z = \left(E_z + \frac{v}{c}H_y\right)\beta \qquad H'_z = \beta\left(H_z - \frac{v}{c}E_y\right)$$

5.8 Show that Maxwell's equations of electro magnetic fields are Lorentz invariant.

Sol. The electric and magnetic field intersities $\overline{E}$ and $\overline{H}$ are corrected by the following Maxwell's equations.

$$\text{div}\,\overline{E} = 4\pi\sigma \qquad (1)$$

$$\text{div}\overline{H} = 0 \qquad (2)$$

$$\text{curl}\,\overline{E} = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad (3)$$

$$\text{curl}\,\overline{H} = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\overline{J} \qquad (4)$$

in Gaussian units.

The cartesian equivalent of (2) is

$$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0 \tag{2a}$$

From (3)

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{1}{c}\frac{\partial}{\partial t}\left(iH_x + jH_y + kH_z\right)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z} &= -\frac{1}{c}\frac{\partial H_x}{\partial t} \\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x} &= -\frac{1}{c}\frac{\partial H_y}{\partial t} \\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y} &= -\frac{1}{c}\frac{\partial H_z}{\partial t} \end{aligned}\right] \tag{3a}$$

Now the transformation of the operators $\overline{E}$ and $\overline{H}$ are

$$\left.\begin{aligned} \frac{\partial}{\partial x'} &= \beta\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right), & \frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z'} &= \frac{\partial}{\partial z}, & \frac{\partial}{\partial t'} &= \beta\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right) \end{aligned}\right] \tag{4}$$

$$\left.\begin{aligned} E'_x &= E_x & H'_x &= H_x \\ E'_y &= \beta\left(E_y-\frac{v}{c}H_z\right) & H'_y &= \beta\left(H_y+\frac{v}{c}E_z\right) \\ E'_z &= \beta\left(E_z+\frac{v}{c}H_y\right) & H'_z &= \beta\left(H_z-\frac{v}{c}E_y\right) \end{aligned}\right] \tag{5}$$

Now,

$$\frac{\partial H'_x}{\partial x'}+\frac{\partial H'_y}{\partial y'}+\frac{\partial H'_z}{\partial z'}=0 \tag{2b}$$

$$\beta\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)H_x+\beta\frac{\partial}{\partial y}\left(H_y+\frac{v}{c}E_z\right)+\beta\frac{\partial}{\partial z}\left(H_z-\frac{v}{c}E_y\right)=0 \tag{6}$$

$$\text{or, } \left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)-\frac{v}{c}\left\{-\frac{1}{c}\frac{\partial H_x}{\partial t}+\left(\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y}\right)\right\}=0$$

$$\text{or, } \left(P-\frac{v}{c}Q\right)=0 \qquad (7)$$

where $$P=\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}$$

$$Q=-\frac{1}{c}\frac{\partial H_x}{\partial t}+\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y}$$

Also from (3a), we get (for S′-frame)

$$-\frac{1}{c}\frac{\partial H_x'}{\partial t'}+\frac{\partial E_y'}{\partial z'}-\frac{\partial E_z'}{\partial y'}=0$$

$$\text{or, } -\frac{1}{c}\beta\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)H_x+\beta\frac{\partial}{\partial z}\left(E_y-\frac{v}{c}H_z\right)$$

$$-\beta\frac{\partial}{\partial y}\left(E_z+\frac{v}{c}H_y\right)=0 \qquad \text{[using (4) \& (5)]}$$

$$\text{or, } \left(-\frac{1}{c}\frac{\partial H_x}{\partial t}+\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y}\right)-\frac{v}{c}\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)=0$$

$$\therefore \left(Q-\frac{v}{c}P\right)=0 \qquad (8)$$

From (8) $Q=\frac{v}{c}P$. Putting this value of Q in (7) we get

$$P-\frac{v}{c^2}P=0$$

$$\Rightarrow P\left(1-\frac{v^2}{c^2}\right)=0$$

$$\Rightarrow P=0 \Rightarrow \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}=0$$

$$\text{and } Q=0 \Rightarrow -\frac{1}{c}\frac{\partial H_x}{\partial t}+\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=0$$

The first is equivalent to

$$\text{div}\,\overline{H} = 0$$

with two similar results of the second equation we can write

$$\text{curl}\,\overline{E} = -\frac{1}{c}\frac{\partial \overline{H}}{\partial t}$$

which are two Maxwell's equations. Since there are two independent equations in the Maxwell's set of equations so we can show the other two equations to be Lorentz invariant.

Hence the set of Maxwell's equations is Lorentz invariant.

5.9 Show that the following are invariant under Lorentz transformation.

(1) $E^2 - H^2$ (2) E.H.

Sol. Lorentz transformation for an observer S′ who is moving with velocity 'v' w.r.t. an observer S is

$$x' = \beta(x - vt)\ ,\ y' = y,\ z' = z.$$

$$t' = \beta\left(t - \frac{vx}{c^2}\right)$$

where

$$\beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Taking the velocity of light to be unity i.e. c = 1, we have,

$$E_x = E_x', \qquad E_y = \beta(E_y' + vH_z'), \qquad E_z = \beta(E_z' + vH_y')$$

$$H_x = H_x', \qquad H_y = \beta(H_y' - vE_z'), \qquad H_z = \beta(E_z' + vE_y')$$

Now,

$$(1)\ E^2 - H^2 = E.E - H.H$$

$$= (E_x^2 + E_y^2 + E_z^2) - (H_x^2 + H_y^2 + H_z^2)$$

$$= (E_x^2 - H_x^2) + (E_y^2 - H_y^2) + (E_z^2 - H_z^2)$$

$$\Rightarrow E^2 - H^2 = (E_x'^2 - H_x'^2) + \beta^2\left[(E_y' + vH_z')^2 - (H_y' - vE_z')^2\right] + \beta^2\left[(E_z' - vH_y')^2 - (H_z' + vE_y')^2\right]$$

$$= (E_x'^2 - H_x'^2) + \beta^2[E_y' + 2vE_y'H_z' + v^2H_z'^2 - H_y'^2 + 2vH_y'E_z' - v^2E_z'^2 + E_z'^2 - 2vE_z'H_y' + vH_y'^2$$

$$-H_z'^2 - 2vH_z'^2E_y' - v^2E_y'^2 \;]$$

$$= (E_x'^2 - H_x'^2) + \beta^2[E_y'^2(1-v^2) - H_z'^2(1-v^2) - H_y'^2(1-v^2) + E_z'^2(1-v^2)$$

$$= (E_x'^2 - H_x'^2) + \beta^2(1-v^2)[(E_y'^2 - H_y'^2) + (E_z'^2 - H_z'^2)\;]$$

$$= (E_x'^2 - H_x'^2) + (E_y'^2 - H_y'^2) + (E_z'^2 - H_z'^2) \qquad \left[\because \beta^2 = \frac{1}{1-v^2}\right]$$

$$= (E_x'^2 + E_y'^2 + E_z'^2) - (H_x'^2 + H_y'^2 + H_z'^2)$$

$$= E'^2 - H'^2$$

(2) $E.H = E_x H_x + E_y H_y + E_z H_z$

$$= E_x'H_x' + \beta(E_y' + vH_z').\beta(H_y' - vE_z') + \beta(E_z' - vH_y').\beta(H_z' + vE_y')$$

$$= E_x'H_x' + \beta^2[E_y'H_y' - vE_y'E_z' + vH_y'H_z' - v^2E_z'H_z' + E_z'H_z' + vE_y'E_z' - vH_y'H_z' - vE_y'H_y']$$

$$= E_x'H_x' + \beta^2[E_y'H_y'(1-v^2) - E_z'H_z'(1-v^2)]$$

$$= E_x'H_x' + \beta^2(1-v^2)(E_y'H_y' + E_z'H_z') = E_x'H_x' + E_y'H_y' + E_z'H_z' = E'H'$$

5.10 Show that D' Atemberts operator $\Box$ is Lorentz invariant.

Sol. Let S and S′ be two inertial frames of reference where S′ is moving with uniform velocity 'υ' relative to S along x-axis.

Now D' Alemberts operator in S fram is

$$\Box^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

and that is S′-frame is

$$\Box'^2 \equiv \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

Lorentz transformation equations are

$$x' = \beta(x - vt), \qquad y' = y, \qquad z' = z, \qquad t = \beta\left(t' + \frac{vx'}{c^2}\right)$$

Now the transformation relation of differential operators in connection to Lorentz transformation are

$$\frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right), \qquad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y},$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial t'} = \beta\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)$$

Now,

$$\frac{\partial^2}{\partial x'^2} = \frac{\partial}{\partial x'}\left(\frac{\partial}{\partial x'}\right)$$

$$= \frac{\partial}{\partial x'}\left[\beta\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\right] = \beta^2\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$$= \beta^2\left(\frac{\partial^2}{\partial x^2} + \frac{v}{c^2}\frac{\partial^2}{\partial x\partial t} + \frac{v}{c^2}\frac{\partial^2}{\partial t\partial x} + \frac{\partial^2}{c^4}\frac{\partial^2}{\partial t^2}\right)$$

$$= \beta^2\left(\frac{\partial^2}{\partial x^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x\partial t} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2}\right)$$

and

$$\frac{\partial^2}{\partial y'^2} = \frac{\partial^2}{\partial y^2}, \qquad \frac{\partial^2}{\partial z'^2} = \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial^2}{\partial t'^2} = \beta^2\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)$$

$$= \beta^2\left(\frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x\partial t} + v^2\frac{\partial^2}{\partial x^2}\right)$$

Now,

$$\Box'^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

$$= \beta^2\left(\frac{\partial^2}{\partial x^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x\partial t} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2}\right) + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\beta^2\left(\frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x\partial t} + v^2\frac{\partial^2}{\partial x^2}\right)$$

$$=\beta^2\left(1-\frac{v^2}{c^2}\right)\frac{v^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\beta^2}{c^2}\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2}{\partial t^2}$$

$$=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad \left(\because \beta^2=\frac{1}{1-\frac{v^2}{c^2}}\right)$$

$$\therefore \Box'^2=\Box^2$$

$$\Rightarrow \Box'=\Box$$

Hence D'Alemberts operator $\Box$ is invariant.

5.11 Lorentz transformation of space and time in four vector form

Consider two systems S and S′, the latter moving with velocity υ relative to former along positive direction of X axis. In Minkowski space let the co-ordinate of an event by represented by the quantities (x, y, z, ict) or x_μ (μ = 1, 2, 3, 4) where

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = ict \tag{1}$$

Lorentz transformations of space and time are written as

$$x'=\frac{x-vt}{\sqrt{1-\beta^2}} \qquad \text{where} \qquad \beta=\frac{v}{c}$$

$$y'=y, \qquad z'=z, \qquad \text{and} \qquad t'=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\beta^2}}$$

or equivalently

$$x_1'=\frac{x_1-vt}{\sqrt{1-\beta^2}}=\gamma\left(x_1+i\beta x_4\right), \qquad x_2'=x_2,$$

$$x_3'=x_3 \qquad \text{and} \qquad x_4'=\frac{ic\left(t-\frac{vx_1}{c^2}\right)}{\sqrt{1-\beta^2}}=\gamma\left(x_4-i\beta x_1\right) \tag{2}$$

Above transformation can be written as

$$
\left.\begin{aligned}
x_1' &= \gamma x_1 + 0.x_2 + 0.x_3 + i\beta\gamma.x_4 \\
x_2' &= 0.x_1 + 1.x_2 + 0.x_3 + 0.x_4 \\
x_3' &= 0.x_1 + 0.x_2 + 1.x_3 + 0.x_4 \\
x_4' &= -i\beta\gamma x_1 + 0.x_2 + 0.x_3 + \gamma x_4
\end{aligned}\right] \qquad (3)
$$

This results can be written compactly by a single equation.

$$x_\mu' = \alpha_{\mu\upsilon} x_\gamma \qquad (4)$$

where

$$\alpha_{\mu\gamma} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \qquad (5)$$

Equation (4) represents the Lorentz transformations of space and time in four vector form.

Similarly Lorentz transformations of any four vector $A_\mu (\mu = 1,2,3,4)$ may be expressed as. $A_\mu' = \alpha_{\mu\gamma} A_\gamma$

5.12 Transformations for charge and current densities.

In relativistic physics the current density and the charge density can not be distinct and completely separable entities, because a charge distribution static in one inertial reference frame will appear as current distribution in the other moving inertial frame. In Minkowski space the charge density ρ and the current density j are grouped together by current four vector j or j_μ according to

$$j_\mu = (j, ic\rho)$$

which may be seen as follows :

Consider a volume element $\quad d\tau = dx_1 dx_2 dx_3$

Then the charge contained in this small volume is

$$dq = \rho dx_1 dx_2 dx_3 \qquad (1)$$

Multiplying above equation by a four vector dx_μ, we get

$$dq dx_\mu = \rho dx_\mu dx_1 dx_2 dx_3 = \rho \frac{dx_\mu}{dt} dx_1 dx_2 dx_3 dt \qquad (2)$$

As charge is invariant (i.e.a scalar), the L.H.S. of equation (2) is a four vector, so R.H.S. of this equation must be a four vector.

Further we know that four dimensional volume element is invariant

i.e. $dx_1 dx_2 dx_3 dx_4$ is invariant

or $dx_1 dx_2 dx_3$ (icdt) is invariant

or $dx_1 dx_2 dx_3\, dt$ is invariant (or scalar)

Therefore for R.H.S of equation (2) to be a four vector, $\rho \frac{dx_\mu}{dt}$ must be a four vector. Let this four vector be represented by

$$j_\mu = \rho \frac{dx_\mu}{dt} \tag{3}$$

So that

$$\left.\begin{aligned} j_1 &= \rho \frac{dx_1}{dt} = \rho x_1 \\ j_2 &= \rho \frac{dx_2}{dt} = \rho x_2 \\ j_3 &= \rho \frac{dx_3}{dt} = \rho x_3 \\ j_4 &= \rho \frac{dx_4}{dt} = \rho \frac{d(ict)}{dt} = ic\rho \end{aligned}\right] \tag{4}$$

i.e. the four vector j_μ, known as current four vector, is represented as

$$j_\mu = (j, ic\rho) \tag{5}$$

As j_μ has been specified as a four vector, it must transform from one inertial frame S to the other inertial frame S′, moving with velocity v relative to S along x_1 axis under Lorentz transformation as

$$j'_\mu = \alpha_{\mu\gamma} j_\mu \tag{6}$$

where

$$\alpha_{\mu\gamma} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{7}$$

So that

$$j_1' = \alpha_{1\gamma} j_\gamma = \alpha_{11} j_1 + \alpha_{12} j_2 + \alpha_{13} j_3 + \alpha_{14} j_4$$

$$= \gamma j_1 + 0.\, j_2 + 0.\, j_3 + i\beta\gamma j_4$$

$$= \gamma\left[j_1 + i\frac{v}{c}(i\rho c) \right]$$

$$= \frac{j_1 - v\rho}{\sqrt{1-\beta^2}} \qquad \text{where} \qquad \beta = \frac{v}{c} \tag{8}$$

$$j_2' = \alpha_{2\upsilon} j_\upsilon = \alpha_{21} j_1 + \alpha_{22} j_2 + \alpha_{23} j_3 + \alpha_{24} j_4$$

$$= 0.\, j_1 + 1.\, j_2 + 0.\, j_3 + 0.\, j_4 = j_2 \tag{9}$$

$$j_3' = \alpha_{3\upsilon} j_\upsilon = \alpha_{31} j_1 + \alpha_{32} j_2 + \alpha_{33} j_3 + \alpha_{34} j_4$$

$$= 0.\, j_1 + 0.\, j_2 + 1.\, j_3 + 0.\, j_4 = j_3 \tag{10}$$

and

$$j_4' = \alpha_{4\upsilon} j_\upsilon = \alpha_{41} j_1 + \alpha_{42} j_2 + \alpha_{43} j_3 + \alpha_{44} j_4$$

$$= -i\beta\gamma j_1 + 0.\, j_2 + 0.\, j_3 + \gamma .\, j_4$$

$$= \gamma(j_4 - i\beta j_1) \tag{11}$$

or, $$ic\rho' = \gamma\left(ic\rho - \frac{i\upsilon}{c} j_1 \right)$$

i.e. $$\rho' = \frac{\rho - \frac{v}{c^2} j_1}{\sqrt{1-\beta^2}} \tag{12}$$

Thus the transformation equations for current and charge densities are

$$j_1' = \frac{j_1 - v\rho}{\sqrt{1-\beta^2}}, \qquad j_2' = j_2, \qquad j_3' = j_3$$

$$\text{and} \qquad \rho' = \frac{\rho - \frac{v}{c^2} j_1}{\sqrt{1-\beta^2}} \qquad (13)$$

The inverse transformations for current and charge densities are obtained by replacing v by –v and interchanging prime and unprime quantities i.e.

$$j_1 = \frac{j_1' + v\rho}{\sqrt{1-\beta^2}}, \qquad j_2 = j_2', \qquad j_3 = j_3'$$

$$\text{and} \qquad \rho = \frac{\rho' + \frac{v}{c^2} j_1'}{\sqrt{1-\beta^2}} \qquad (14)$$

Discussion of Results :

(a) Equation of Continuity in Covariant form :

The continuity equation is

$$\text{div} j + \frac{\partial \rho}{\partial t} = 0$$

This can be written as

$$\text{div} j + \frac{\partial (ic\rho)}{\partial (ict)} = 0$$

$$\Rightarrow \frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{\partial j_4}{\partial x_4} = 0 \qquad (\text{Since } j_4 = ic\rho \text{ and } x_4 = ict)$$

$$\text{i.e. } \frac{\partial j_4}{\partial x_4} = \Box j_4 = 0 \qquad (15)$$

where $\Box j_4 \frac{\partial}{\partial x_\mu} =$ is the four dimensional divergence operator.

Equation (15) is covariant equation i.e. its form is unaltered under Lorentz transformations. This equation expresses that the four divergence of the current four vector j_μ vanishes.

(b) Special case : When charge distribution is at rest in system S, then j = 0. Therefore the transformation equations (13) take the form

$$j'_1 = -v\rho, \qquad j'_2 = 0, \qquad j'_3 = 0$$

and $$\rho' = \frac{\rho}{\sqrt{1-\beta^2}} \tag{16}$$

(c) Invariance of Charge :

If $d\tau' = dx'_1\, dx'_2\, dx'_3$ is the volume element in system S′, then the charge contained in this volume in system S′ is

$$d\tau' = dx'_1\, dx'_2\, dx'_3 = \left(\frac{\rho}{\sqrt{1-\beta^2}}\right)\left(dx_1\sqrt{1-\beta^2}\right)dx_2 dx_3$$

$$= \rho dx_1\, dx_2\, dx_3 = dq$$

i.e. charge measured in system S′ is the same as that in system S. i.e., electric charge is invariant under Lorentz transformations.

As $\rho^1 = \frac{\rho}{\sqrt{1-\beta^2}}$; the electric charge density is not relativistically invariant.

5.13 To prove that in the case of free space $\Box^2\, \bar{E} = 0 = \Box^2\, \bar{H}$.

Sol. For free space $\rho = 0$

The Maxwell's equation in free space are

$$\text{div } \bar{E} = 0 \tag{1}$$

$$\text{div } \bar{H} = 0 \tag{2}$$

$$\text{curl } \bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t} \tag{3}$$

$$\text{curl } \bar{H} = -\frac{1}{c}\frac{\partial \bar{E}}{\partial t} \tag{4}$$

Now taking curl of both sides of (3), we get

$$\text{curl curl } \bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\text{curl } \bar{H})$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \bar{E}}{\partial t}\right) \qquad \text{(by (4))}$$

$$= -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad (5)$$

But,

$$\text{grad div } \vec{E} = \nabla^2 \vec{E} + \text{curl curl } \vec{E}$$

$$\Rightarrow \text{curl curl } \vec{E} = \text{grad div } \vec{E} - \nabla^2 \vec{E}$$

$$\therefore (5) \Rightarrow \text{grad (div } \vec{E}) - \nabla^2 \vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \text{grad } (0) - \nabla^2 \vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{[using (1)]}$$

$$\Rightarrow \nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0 \qquad \Rightarrow \Box^2 \vec{E} = 0$$

Again taking curl of both sides of (4) we get,

$$\text{curl curl } \vec{H} = \frac{1}{c}\frac{\partial}{\partial t}(\text{curl } \vec{E})$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \vec{H}}{\partial t}\right) \qquad \text{(using (3))}$$

$$= -\frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} \qquad (6)$$

But, $\quad \text{grad div } \vec{H} = \nabla^2 \vec{H} + \text{curl curl } \vec{H}$

$$\Rightarrow \text{curl curl } \vec{H} = \text{grad div} \vec{H} - \nabla^2 \vec{H}$$

$$\therefore (6) \Rightarrow \text{grad (div } \vec{H}) - \nabla^2 \vec{H} = -\frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{H} - \frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} = \text{grad } (0) \qquad \text{[using (2)]}$$

$$\Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{H} = 0$$

$$\Rightarrow \Box^2 \vec{H} = 0$$

Hence $\quad \Box^2 \vec{E} = 0 = \Box^2 \vec{H}$

5.14 Show that there exists one vector potential $\bar{A}$ and a scalar potential ϕ satisfying Maxwell's equations.

Sol. In Gaussian units, Maxwell's equations are

$$\nabla.\bar{E} = 4\pi\sigma \quad (1)$$

$$\nabla.\bar{H} = 0 \quad (2)$$

$$\nabla \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{H}}{\partial t} \quad (3)$$

$$\nabla \times \bar{H} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t} + \frac{4\pi}{c}\bar{J} \quad (4)$$

From (2),

$$\nabla.\bar{H} = 0$$

$$\therefore \text{div}\,(\text{curl}\,\bar{u}) = 0$$

Therefore $\bar{H}$ = curl (of some vector)

$$\bar{H} = \text{curl}\,\bar{A} \text{ (say)}$$

From (3),

$$\nabla \times \bar{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \bar{A})$$

$$\Rightarrow 0 = \nabla \times \left(-\bar{E} - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}\right) \qquad \because \text{curl (grad } \phi) = 0$$

Therefore,

$$-\bar{E} - \frac{1}{c}\frac{\partial \bar{A}}{\partial t} = \nabla\phi$$

where ϕ is some scalar function. Hence, there exists two potentials $\bar{A}$ (vector potential) and ϕ (Scalar potential) satisfying Maxwell's equations, as they consists of only two independent equations.

●●●

Miscelenous Examples

1. If u and υ are two velocities in the same direction and V their resultant velocity given by

$\tan^{-1}\frac{V}{c} = \tanh^{-1}\frac{u}{c} + \tanh^{-1}\frac{\upsilon}{c}$. then deduce the law of composition of velocities from this equation.

Sol. Given that $\tanh^{-1}\frac{V}{c} = \tanh^{-1}\frac{u}{c} + \tanh^{-1}\frac{v}{c}$

This equation is expressible as

$$\frac{1}{2}\log\frac{c+V}{c-V} = \frac{1}{2}\log\frac{c+u}{c-u} + \frac{1}{2}\log\frac{c+v}{c-v}$$

$$\text{or, } \log\frac{c+V}{c-V} = \log\frac{c+u}{c-u}\cdot\frac{c+v}{c-v}$$

$$\Rightarrow \frac{c+V}{c-V} = \frac{c+u}{c-u}\cdot\frac{c+v}{c-v} = \frac{c^2+(u+v)c+uv}{c^2-(u+v)c+uv}$$

$$\Rightarrow \frac{c+V}{c-V} - 1 = \frac{c^2+(u+v)c+uv}{c^2-(u+v)c+uv} - 1$$

$$\Rightarrow \frac{c+V-c+V}{c-V} = \frac{c^2+(u+v)c+uv-c^2+(u+v)c-uv}{c^2-(u+v)c+uv}$$

$$\Rightarrow \frac{2V}{c-V} = \frac{2(u+v)c}{c^2-(u+v)c+uv}$$

$$\Rightarrow \frac{c-V}{V} = \frac{c^2-(u+v)c+uv}{(u+v)c} \Rightarrow \frac{c}{V} - 1 = \frac{c}{u+v} - 1 + \frac{uv}{c(u+v)}$$

$$\Rightarrow \frac{c}{V} = \frac{c}{u+v} + \frac{uv}{c(u+v)} \Rightarrow \frac{c}{V} = \frac{c^2+uv}{c(u+v)}$$

$$\Rightarrow \frac{V}{c} = \frac{c(u+v)}{c^2+uv} \Rightarrow V = \frac{c^2(u+v)}{c^2+uv}$$

$$\Rightarrow V = \frac{u+v}{1+\frac{uv}{c^2}}$$ [Dividing both numerator and denominator by c^2]

This is the required expression for V.

2. An electron is moving with a speed of .85C in a direction opposite to that of a moving photon. Calculate the relative velocity of electron and photon.

Solution. Suppose that the photon and electron are moving along positive and negative directions of x-axis respectively. Suppose that the electron moving with velocity –.85C is at rest in the system S. Hence the system S′ may be assumed to have velocity .85C relative to S (electron). Thus

$v = .85C,\ u' = c$

Let V be the relative velocity.

Then

$$V = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{.85c + c}{1 + \frac{.85c^2}{c^2}}$$

$$= \frac{c(1+.85)}{(1+.85)} = c.$$

Hence relative velocity of electron & photon is c.

3. The length of a rocket ship is 100 meters on the ground. When it is in flight its length observed on the ground is 99 meters, calculate its speed.

Solution. By the result of Lorentz contraction,

$$l' = l\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

Hence,

$$99 = 100\sqrt{1 - \frac{\upsilon^2}{c^2}} \quad \text{or,} \quad \left(\frac{99}{100}\right)^2 = 1 - \frac{\upsilon^2}{c^2}$$

$$\text{or,} \quad \frac{\upsilon^2}{c^2} = 1 - \frac{9801}{10000} = \frac{10000 - 9801}{10000}$$

$$\text{or,} \quad \frac{\upsilon^2}{c^2} = \frac{199}{10000} \quad \text{or,} \quad \frac{\upsilon}{c} = \frac{\sqrt{199}}{100}$$

$$\text{or,} \quad \upsilon = \frac{\sqrt{199}}{100} \times 3 \times 10^8\,\text{m/s} \left[\because c = 3 \times 10^8\,\text{m/s}\right]$$

$$\text{or,} \quad \upsilon = \sqrt{199} \times 3 \times 10^6\,\text{m/s} \quad \text{or,} \quad \upsilon = 42.3 \times 10^6\,\text{m/s}.$$

4. At what speed should a clock be moved so that it may appear to lose 1 minute in each hour.

Solution. Since the clock is to lose one minute in one hour. Hence the moving clock records 59 minutes for each hour recorded by clock stationary relative to the observer. Let 'υ' be the required speed of the moving clock.

Given,

$$\Delta t' = 59 \text{ minutes}, \qquad \Delta t = 60 \text{ minutes.}$$

Evidently $\Delta t > \Delta t'$

Hence

$$\Delta t = \beta \Delta t' = \frac{\Delta t'}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

or, $\left[1-\frac{\upsilon^2}{c^2}\right]^{\frac{1}{2}} \Delta t = \Delta t'$ or, $\left(1-\frac{\upsilon^2}{c^2}\right)^{\frac{1}{2}} .60 = 59$ or, $\left(1-\frac{\upsilon^2}{c^2}\right)^{\frac{1}{2}} = \frac{59}{60}$

or, $1-\frac{\upsilon^2}{c^2} = \left(\frac{59}{60}\right)^2$ or, $\frac{\upsilon^2}{c^2} = 1-\left(\frac{59}{60}\right)^2$ or, $\frac{\upsilon^2}{c^2} = 1-\frac{3481}{3600}$

or, $\frac{\upsilon^2}{c^2} = \frac{3600-3481}{3600}$ or, $\upsilon^2 = c^2 \frac{119}{3600}$ or, $\upsilon = c\frac{\sqrt{119}}{60}$

or, $\upsilon = 0.18C$

$$\text{or, } \upsilon = 5.4\times10^{10} \text{ cm/s} \ \left(c = 3\times10^{10}\text{cm/s}\right)$$

5. Prove that four dimensional volume element dxdydzdt is invariant under Lorentz transformations.

Solution. By Lorentz contraction, we have

$$dx' = \sqrt{1-\frac{\upsilon^2}{c^2}}\, dx,\ dy = dy'$$

$$dz = dz'$$

Also by time dilation,

$$dt' = \frac{dt}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$\therefore dx'.dy'.dz'.dt' = dx\sqrt{1-\frac{\upsilon^2}{c^2}}\, dydz \frac{dt}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$$

$$= dx.dy.dz.dt$$

or, $dx'.dy'.dz'.dt' = dx.dy.dz.dt$ This proves the required result.

6. Show that $x^2 + y^2 + z^2 - c^2t^2$ is Lorentz invariant.

Solution. By Lorentz transformation equations,

$$x' = \beta(x - \upsilon t), \qquad y' = y, \qquad z' = z,$$

$$t' = \beta\left(t - \frac{\upsilon x}{c^2}\right) \text{ where } \beta = \frac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

Now,

$$x'^2 + y'^2 + z'^2 - c'^2t'^2$$

$$= \beta^2(x - \upsilon t)^2 + y^2 + z^2 - c^2\beta^2\left(t - \frac{\upsilon x}{c}\right)^2$$

$$= \beta^2\left[x^2 + \upsilon^2t^2 - 2\upsilon xt - c^2\left(t^2 + \frac{\upsilon^2x^2}{c^4} - \frac{2\upsilon tx}{c^2}\right) + y^2 + z^2\right]$$

$$= \beta^2\left(1 - \frac{\upsilon^2}{c^2}\right)x^2 - c^2t^2\left(1 - \frac{\upsilon^2}{c^2}\right)\beta^2 + y^2 + z^2$$

$$= x^2 - c^2t^2 + y^2 + z^2 \left[\because \beta^2 = \frac{1}{1 - \frac{\upsilon^2}{c^2}}\right]$$

$$= x^2 + y^2 + z^2 - c^2t^2$$

$$= x'^2 + y'^2 + z'^2 - c'^2t'^2 = x^2 + y^2 + z^2 - c^2t^2.$$

This proves the required result.

7. Show that for low values of 'υ', the Lorentz transformation approaches to Galilean.

Solution. Lorentz transformation equations are

$$x' = \frac{x - \upsilon t}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}, \; y' = y, \; z' = z$$

It 'υ' is small i.e. $\upsilon << c$ so that $\left(\frac{\upsilon}{c}\right) \to 0$ then the foregoing equations become

$x' = x - \upsilon t$, $y' = y$, $z' = z$, $t' = t$ which are Galilean transformation equations.

8. Show that $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (cdt)^2$ is invariant under Lorentz transformations.

Solution. By Lorentz transformation

$$x' = \beta(x - \upsilon t),\ y' = y,\ z' = z,\ t' = \beta\left(t - \frac{\upsilon x}{c^2}\right)$$

$$\text{where } \beta = \frac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

$$\therefore\ dx' = \beta(dx - \upsilon dt),\ dt' = \beta\left(dt - \frac{\upsilon}{c^2}dx\right)$$

$$dy' = dy,\ dz' = dz,\ c' = c.$$

$$(dx')^2 = \beta^2\left[(dx)^2 + \upsilon^2(dt)^2 - 2\upsilon dx dt\right]$$

$$(cdt')^2 = c^2\beta^2\left[(dt)^2 + \frac{\upsilon^2}{c^4}(dx)^2 - \frac{2\upsilon}{c^2}dt dx\right]$$

The last two imply that

$$(dx')^2 - (cdt')^2 = \beta^2\left(1 - \frac{\upsilon^2}{c^2}\right)(dx)^2 - c^2(dt)^2\left(1 - \frac{\upsilon^2}{c^2}\right)\beta^2$$

$$= (dx)^2 - c^2dt^2 \left[\because \beta^2 = \frac{1}{1 - \frac{\upsilon^2}{c^2}}\right] \quad \text{.... (1)}$$

$\therefore$ Also $dy' = dy,\ dz' = dz$. .... (2)

By (1) and (2) we have

$$(dx')^2 + (dy')^2 + (dz')^2 - (cdt')^2 = (dx)^2 + (dy)^2 + (dz)^2 - (cdt)^2$$

This proves $(dx)^2 + (dy)^2 + (dz)^2 - (cdt)^2$ is invariant under Lorentz transformation.

9. How much electric energy could theoritically be obtained by annihilation of 1 gm of matter?

Solution. We know that $E = mc^2$, $m = 1$ gm, $c = 3 \times 10^{10}$ cm/s.

$\therefore\ E = mc^2 = 1 \times (3 \times 10^{10})^2$ ergs $= 9 \times 10^{20}$ ergs

But 1 electron volt $= 1.602 \times 10^{-12}$ ergs.

$$\therefore\ E = 9 \times 10^{20} \text{ ergs} = \frac{9 \times 10^{20}}{1.602 \times 10^{-12}} \text{ ev} = \frac{9 \times 10^{32}}{1.602} \text{ ev} = 5.618 \times 10^{32} \text{ ev}$$

10. An electron and a positron practically at rest come together and annihilate eachother, producing two photons of equal energy. Find the energy of each photon.

Solution. We know that

m_0 = rest . mass of electron = rest . mass of position = 9×10^{-28} gm.

Energy E of each photon given by

$$E = m_0c^2 = 9 \times 10^{-28} \times 9 \times 10^{20} \text{ ergs.}$$

$$= 81 \times 10^{-8} \text{ ergs}$$

$$= \frac{81 \times 10^{-8}}{1.6 \times 10^{-12}} \text{eV} \quad (\because \text{ieV} = 1.6 \times 10^{-12} \text{erg})$$

$$= \frac{81}{1.6} \times 10^4 \text{eV} = 5.602 \times 10^4 \text{eV}$$

11. Calculate the velocity at which the mass of a particle becomes 8 times its rest mass.

Solution. Given that $m = 8m_0$. We have,

$$m = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} \qquad \text{or, } 8m_0 = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} \qquad \text{or, } 8 = \frac{1}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

$$\text{or, } \sqrt{1 - \frac{\upsilon^2}{c^2}} = \frac{1}{8} \qquad \text{or, } 1 - \frac{\upsilon^2}{c^2} = \frac{1}{64} \qquad \text{or, } \frac{\upsilon^2}{c^2} = 1 - \frac{1}{64} = \frac{63}{64}$$

$$\text{or, } \upsilon = \left(\frac{63}{64}\right)^{\frac{1}{2}} c \qquad \text{or, } \upsilon = 0.992C = .992 \times 3 \times 10^{10} \text{cm/s} \qquad \text{or, } \upsilon = 2.976 \times 10^{10} \text{cm/s}$$

12. The rest mass of as electron is 9×10^{-28} gm. What will be mass if it were moving with velocity $\frac{4}{5}$ times the speed of light.

Solution. If 'm' be the mass of an electron, when its velocity is

$$\upsilon = \frac{4c}{5} \text{ so that } \frac{\upsilon^2}{c^2} = \frac{16}{25} \qquad \text{Also gives } m_0 = 9 \times 10^{-28}$$

Now

$$m = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}} = \frac{9 \times 10^{-28}}{\sqrt{1 - \frac{16}{25}}} = \frac{9 \times 10^{-28}}{\sqrt{\frac{9}{25}}} = \frac{5 \times 9 \times 10^{-28}}{3} = 15 \times 10^{-28} \text{ gm.}$$

13. Calculate the kinetic energy of an electron moving with a velocity of 0.98 times the velocity of light in the laboratory system.

Solution. We have, $E = mc^2$ Also $E = T + m_0c^2$

Hence relativistic K.E. 'T' is given by

$$T = E - m_0c^2 = mc^2 - m_0c^2 = (m - m_0)c^2$$

$$= \left[\frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}} - m_0\right]c^2 = m_0c^2\left[\frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} - 1\right]$$

It is given that $\upsilon = 0.98c$ so that $\frac{\upsilon^2}{c^2} = (.98)^2$ with this value, the last becomes

$$T = m_0c^2\left[\frac{1}{\sqrt{1-(.98)^2}} - 1\right] = m_0c^2\left[\frac{1}{.199} - 1\right]$$

$$= \left(\frac{.801}{.199}\right)m_0c^2 = \frac{.801}{199} \times 9 \times 10^{-28} \times \left(3\times10^{10}\right)^2 = 3.26 \times 10^{-6} \text{ ergs.}$$

14. From the relativistic concept of mass and energy, show that the K.E. of the moving mass 'm' with velocity 'υ' is $\frac{m\upsilon_0^2}{2}$ when $\upsilon << c$.

Solution. We know that,

E = K.E of moving particle + energy at rest

or, $mc^2 = T + m_0c^2$ as $E = mc^2$, T = K.E of moving mass.

or, $T = c^2(m - m_0)$

$$= c^2\left[\frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}} - m_0\right] = m_0c^2\left[\frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}} - 1\right] = m_0c^2\left[\left(1-\frac{\upsilon^2}{c^2}\right)^{-\frac{1}{2}} - 1\right] = m_0c^2\left[1+\frac{\upsilon^2}{2c^2} - 1\right]$$

(neglecting higher powers as $\upsilon << c$)

$$= m_0c^2 \cdot \frac{\upsilon^2}{2c^2} = \frac{1}{2}m_0\upsilon^2 \qquad \text{Hence } T = \frac{1}{2}m_0\upsilon^2$$

MULTIPLE CHOICE QUESTIONS

1. The value of the speed of light c is

a) 3×10^{10} cm/s b) 3×10^{8} cm/s c) 3×10^{-10} cm/s d) 3×10^{-8} cm/s

Ans. a) 3×10^{10} cm/s

2. Which of the following is not a Lorentz transformation?

a) $x' = \beta(x - \upsilon t)$ b) $y' = y$ c) $z' = z$ d) $t' = \left(t - \frac{\upsilon x}{c^2}\right)$

Ans. e) $\gamma = \mu(x - t)$

3. The value of β is given by

a) $\beta = \frac{1}{\sqrt{1+\frac{\upsilon^2}{c^2}}}$ b) $\beta = \frac{1}{\sqrt{1+\frac{c^2}{\upsilon^2}}}$ c) $\beta = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ d) $\beta = \frac{1}{\sqrt{1-\frac{c^2}{v^2}}}$

Ans. c) $\beta = \frac{1}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$

4. Lorentz transformation reduces to Galilean transformation under the condition.

a) $\upsilon > c$ b) $\upsilon < c$ c) $\upsilon >> c$ d) $\upsilon << c$

Ans. d) $\upsilon << c$

5. Speed of light is a

a) Constant b) Variable c) Depends on the motion of the source of light

Ans. a) Constant

6. The result of two successive Lorentz transformation is a

a) Lorentz transformation b) Galilean transformation

c) Not a transformation d) Both Lorentz and Galilean transformation.

Ans. a) Lorentz transformation

7. Lorentz contraction is given by

a) $l' = l\sqrt{1+\frac{\upsilon^2}{c^2}}$ b) $l' = l\sqrt{1-\frac{\upsilon^2}{c^2}}$ c) $l' = l\sqrt{1+\frac{c^2}{\upsilon^2}}$ d) $l' = l\sqrt{1-\frac{c^2}{\upsilon^2}}$

Ans. b) $l' = l\sqrt{1-\frac{\upsilon^2}{c^2}}$

8. Two events are said to be simultaneous if they occur at the

a) Same time   b) Different time

c) Any time

Ans. a) Same time

9. Time dilation is given by

a) $\Delta t' = \frac{\Delta t}{\sqrt{1+\frac{\upsilon^2}{c^2}}}$ b) $\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ c) $\Delta t' = \frac{\Delta t}{\sqrt{1+\frac{c^2}{\upsilon^2}}}$ d) $\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{c^2}{\upsilon^2}}}$

Ans. b) $\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$

10. Lorentz transformation form a

a) Group   b) Subgroup   c) Normal Subgroup

Ans. a) Group

11. Variation of mass with velocity is given by

a) $m = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ b) $m = \frac{m_0}{\sqrt{1+\frac{\upsilon^2}{c^2}}}$ c) $m = m_o\sqrt{1-\frac{\upsilon^2}{c^2}}$ d) $m = m_0\sqrt{1+\frac{\upsilon^2}{c^2}}$

(m_0 = rest mass)

Ans. a) $m = \frac{m_0}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$

12. Equivalence of mass and energy is given by

a) $E = mc^2$ b) $E = m^2c$ c) $E = \frac{m}{c^2}$ d) $E = \frac{m^2}{c}$

Ans. a) $E = mc^2$

13. Transformation formula for mass is given by along x-direction,

a) $m' = \frac{m\left(1+\frac{\upsilon u_x}{c^2}\right)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ b) $m' = \frac{m\left(1-\frac{\upsilon u_x}{c^2}\right)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$ c) $m' = \frac{m\left(1+\frac{\upsilon u_x}{c^2}\right)}{\sqrt{1+\frac{\upsilon^2}{c^2}}}$ d) $m' = \frac{m\left(1-\frac{\upsilon u_x}{c^2}\right)}{\sqrt{1+\frac{\upsilon^2}{c^2}}}$

Ans. b) $m' = \frac{m\left(1-\frac{\upsilon u_x}{c^2}\right)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}$

14. $p^2 - \frac{E^2}{c^2}$ is a) Lorentz invariant b) Galilean invariant c) net invariant

Ans. a) Lorentz invariant

15. Maxwell's equations are invariant under

a) Lorentz transformation b) Galilean transformation c) Any kind of transformation.

Ans. a) Lorentz transformation

16. $E^2 - H^2$ is

a) Lorentz invariant b) Galilean invariant c) Maxwell invariant d) Not invariant.

Ans. a) Lorentz invariant

17. $\square^2$ is invariant under

a) Lorentz transformation b) Galilean transformation c) Maxwell transformation

Ans. Lorentz transformation

18. In case of free space

a) $\square^2 E = 0$ b) $\square^2 E \neq 0$ c) $\square^2 E > 0$ d) $\square^2 E < 0$

19. E and H remains unchanged under

a) Lorentz transformation b) Galilean transformation c) Gauge transformation

Ans. c) Gauge transformation

20. E.H is invariant under

a) Lorentz transformation b) Maxwell transformation

c) Gauge transformation d) Any transformation

Ans. a) Lorentz transformation

GAUHATI UNIVERSITY
QUESTION PAPERS
YEAR - 2002

1. What are the circumstances that prompted Einstein to formulate the two postulates of special theory of relativity?

Sol. Done earlier

Hints : Galilias principle of relativity with the three experiments.

(1) From the observation of light coming from the binary stars.

(2) Fizeu's experiment. } (Deduce all the three)

(3) Michelson-Morley experiment.

2. Derive the relativistic addition law of velocities. Hence find the speed of one electron relative to another electron which are separated from the same sample of substance with equal speed 0.66c in opposite directions.

Sol. **1st part :** Derive.

$$u_x = \frac{u'_x + \upsilon}{1 + \frac{\upsilon u_x}{c^2}}, \quad u_y = \frac{u'_y}{\beta\left(1 + \frac{\upsilon}{c^2} u'_x\right)}, \quad u_z = \frac{u'_z}{\beta\left(1 + \frac{\upsilon u'_x}{c^2}\right)}$$

Done earlier

2nd part : Proceed as example (2) of miscellenous examples.

Hint : $\upsilon = .66c$, $u' = c$

Let V be the relative velocity.

$$\text{Then } V = \frac{u' + \upsilon}{1 + \frac{u'\upsilon}{c^2}} = \frac{.66c + c}{1 + \frac{.66c^2}{c^2}} = \frac{(.66+1)c}{(1+.66)} = c.$$

3. State consequences of Lorentz transformation. Discuss (1) length contraction (2) time dilation.

Sol. Statement of consequences of Lorentz transformation

(1) Lorentz-Filzerald contraction. (2) Time dilation.

(3) It is impossible to send a signal with velocity greater than the velocity of light c.

2nd part : Discuss Lorentz-Filzerald contraction and time dilation.

4. Prove mathematically by three different approaches that the speed of light c is invariant.

Sol. Dissscuss the following as done earlier.

(1) From the observation of light coming from the binary stars.

(2) Fizeu's experiment.

(3) Michelson-Morley experiment.

5. Establish the formula

$$m = \frac{m_0}{\sqrt{1 - \frac{\upsilon^2}{c^2}}}$$

Hence calculate the velocity at which the mass of a moving particle becomes 8 times its rest mass.

Sol. **1st part :** Done earlier.

Hints : Deduce variation of mass upto equation (4).

2nd part : Done in miscellenous example.

Hints : Question number (11).

6. Devine the expressions for four-velocity and four-force.

Sol. Done earlier.

Hints : Deduce the equations

$$u_1 = \frac{u_x}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad u_2 = \frac{u_y}{\sqrt{1 - \frac{u^2}{c^2}}} \quad u_3 = \frac{u_z}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad u_4 = \frac{ic}{\sqrt{1 - \frac{u^2}{c^2}}}$$

7. Establish the transformation equations for $\overline{E}$ (E_x, E_y, E_z) and $\overline{H}$ (H_x, H_y, H_z), the electric and magnetic field intensities. Hence prove that $\overline{E}^2 - \overline{H}^2$ and $\overline{E}.\overline{H}$ are Lorentz invariant.

Sol. Done earlier.

Hints : Deduce the equations $E'_x = E_x$, $\quad H'_x = H_x$,

$$E'_y = \beta\left(E_y - \frac{\upsilon}{c}H_z\right), \quad H'_y = \beta\left(H_y + \frac{\upsilon}{c}E_z\right), \quad E'_z = \beta\left(E_z + \frac{\upsilon}{c}H_y\right) \quad , \; H'_z = \beta\left(H_z - \frac{\upsilon}{c}E_y\right)$$

and then show that $E^2 - H^2 = E'^2 - H'^2$ and $E.H = E'. H'$

8. Explain the background how to deduce Maxewell's equations. Hence, show that all the four Maxwell's equations are not independent.

Sol. Done earlier.

Hints : Write all as given in Maxwell's equation with deduction.

GAUHATI UNIVERSITY
QUESTION PAPERS
YEAR - 2003

1. State the two pastcelates of special theory of relativity. Hence derive the Lorentz transformation equations.

Sol. Postulates are as follows :

(1) All laws of physics (excluding gravity) must be invariant w.r.t. observers moving in relative uniform motion.

(2) The speed of light c(in vacuum) is constant w.r.t. observers moving in relative uniform motion.

2nd part : Done earlier

Hints : Deduce the equations

$$x' = x\,(x - \upsilon t)$$

$$y' = y\ ,\ z' = z$$

$$t' = z\left(t - \frac{\upsilon x}{c^2}\right)$$

2. Define space-like and time like intervals of special theory of relativity. Examine these intervals validity for point-events of space time continuum.

Sol. Done earlier..

Hints : Deduce all as given in space like and time like interval.

3. The length of a rocket-ship is 100 metres on the ground. When it is in flight its length observed on the ground is 99 metres, calculate its speed.

Sol. Done in miscellenous example.

Hints : Question number (3)

$$\upsilon = 42.3 \times 10^6 \text{ m/s}$$

4. At what speed should a clock be moved so that it may appear to lose 1 minute in each hour?

Sol. Done in miscellenous example

Hints : Question number (4)

$$\upsilon = 5.4 \times 10^{10} \text{ m/s}$$

5. Establish the formula

$$m' = \frac{m\left(1-\frac{\upsilon u x}{c^2}\right)}{\sqrt{1-\frac{\upsilon^2}{c^2}}}.$$

Hence formulate the energy momentum vector of special theory of relativity and show that it is Lorentz invariant.

Sol. **1st part :** Deduce transformation law of mass.

2nd part : Deduce upto

$$\frac{E'^2}{c^2} - p'^2 = \frac{E^2}{c^2} - p^2$$

All are done earlier.

6. Deduce the Einstein's equation of energy

$$E = mc^2.$$

Sol. Done earlier..

Hints : Equivalent of mass and energy.

7. A body of mass M disintegrates while at rest into two parts of rest masses M_1 and M_2, show that the energies E_1 & E_2 of the parts are

$$E_1 = \frac{c^2}{2M}\left(M^2 + M_1^2 - M_2^2\right)$$

$$E_2 = \frac{c^2}{2M}\left(M^2 + M_2^2 - M_1^2\right)$$

Sol. Done in chapter 2.

8. Show that the Maxwell's equations of electromagnetic fields are Lorentz invariant.

Sol. Done earlier.

9. **Write down Maxwell's equations in Gaussian units. Show that there exists a vector potential $\overline{A}$ and a scalar potential ϕ satisfying these equations. Hence show that the d' Alembertis operator $\Box^2$ is Lorentz invariant.**

Sol. **Done earlier.**

Hints : Deduce $\Box'^2 = \Box^2$

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