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M.A./M.Sc. in Mathematics

Semester 3

Paper IV

Space Dynamics

Contents:

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Unit-1

Basic formula of a spherical triangle :

1.1 The two body problem :

The motion of two body problem means the motion of one body of mass m_1 (say) relative to another mass m_2 (say) due to the mutual gravitational attraction. The motion of the planet relative to the sun in our solar system is the classical example of two body problem. But the planetary motion is governed by Tohann Kepler (1571-1630). In this case the force of attraction is the body of force of consideration arising out of the two masses in which the persence of other masses (other planets) are neglected. According to the Keplerian laws the planet move in elliptic orbit but when we take into account of the presence of other planets (or masses), the elliptic motion will be disturbed giving rise to "disturbed elliptic motion" otherwise perturbative motion. Since the sum's motion is 700 times the total mass of all the planets so the pertubation is ignorable. The motion of an artificial stallite relative to the earth can also be termed as two-body problem as the man of the satellite is very small compred to the man of the earth. The idea of the satellites motion is derived from the motion of the moon round the earth. Therefore the motion of satellite, we must have the clear understanding of the motion of a planet governed by the following there law's of Kepler.—

(a) Every planet moves round the sun in an elliptic orbit with the sun at one of its foci.

(b) The radius vector joining the planet and the sun sweeps out equal areas in equal interval of time.

(c) The square of the time period (i.e time of one complete revolution) is proportional to the cube of the semi major axis of the elliptic path.

viz – $T^2 \propto a^3$ i.e. $T^2 = \left(\frac{4t}{h}\right) a^3$.

1.2. Spherical Triangle :

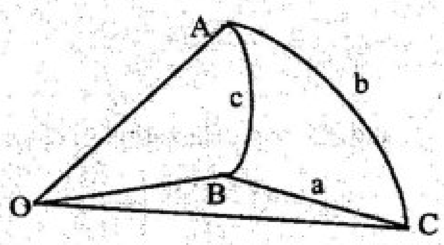
The great circular planes when cut a sphere then a shape of a triangle with curves as sideds is farmed. This is called a spherical triangle.

In figure Δ ABC is a spherical triangle.

O is the centre of the sphere.

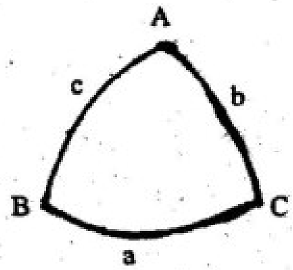
In a spherical triangle Δ ABC,

the following formula can be deducted—



(a) $\cos a = \cos b \cos c + \sin b \sin c \cos A$

(b) $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$



(c) cos (inner side) $\times$ cos (inner angle)

= sin (inner side) $\times$ cot (other angle)

– sin (inner side) $\times$ cot (other angle)

i.e. $\cos a \cos C = \sin a \cot B - \sin c \cot B$

(d) $\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$ etc.

(sine – cosine formula)

1.3. The motion of the centre of mass :

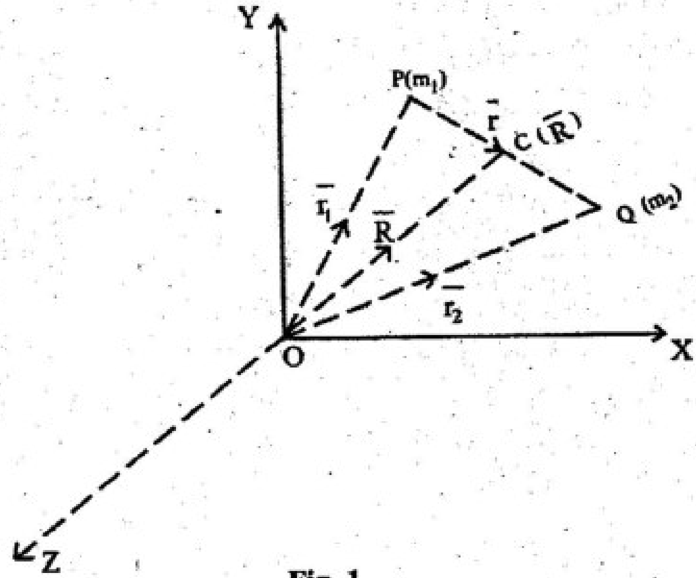


Fig. 1.

let m_1 and m_2 be the two masses at P and Q and $\overline{r_1}$ and $\overline{r_2}$ be their position vectors with respect to intertial frame of reference O – xyz. Let $\overline{R}$ be the position vector of their centre of mass place d at c and $\bar{r}$ be the position vector of m_2 relative to m_1 then

$$\overline{r_1} + \bar{r} = \overline{r_2} \rightarrow (1)$$

If $\overline{f_{12}}$ is acceleraation of mass m_1 due to the mass m_2 then

$$m_1 \overline{f_{12}} = G \frac{m_1 m_2}{r^2} \left(\frac{\bar{r}}{r}\right)$$

and $\overline{f_{21}}$ is acceleraation of mass m_2 due to the mass m_1 then

$$m_2 \overline{f_{21}} = G \frac{m_2 m_1}{r^2} \left(-\frac{\bar{r}}{r}\right)$$

due to Newton's law of gravity.

$\therefore$ The eq^n of motions of the two mass are

$$m_1\ddot{\bar{r}}_1 = G\frac{m_1m_2}{r^2}\left(\frac{\bar{r}}{r}\right)$$

$$= G\frac{m_1m_2}{r^3}\,\bar{r} \;\rightarrow (2)$$

and

$$m_2\ddot{\bar{r}}_2 = G\frac{m_1m_2}{r^2}\left(-\frac{\bar{r}}{r}\right)$$

$$= -G\frac{m_1m_2}{r^3}\,\bar{r} \rightarrow (3)$$

Adding (2) and (3), we get

$$m_1\ddot{\bar{r}}_1 + m_2\ddot{\bar{r}}_2 = 0$$

Integrating twice successively, we get

$$m_1\bar{r}_1 + m_2\bar{r}_2 = \bar{c}_1\, t + \bar{c}_2 \quad \rightarrow (4)$$

$$\text{But } \bar{R} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1 + m_2}$$

$$\text{or } M\bar{R} = m_1\bar{r}_1 + m_2\bar{r}_2$$

Where $m_1 + m_2 = M$

using this result in (4), we get

$$M\bar{R} = \bar{c}_1\, t + \bar{c}_2$$

$$\text{or } \bar{R} = \left(\frac{\bar{c}_1}{M}\right)t + \left(\frac{\bar{c}_2}{M}\right) \rightarrow (5)$$

Which show that centre of mass of the two masses is either at rest or of unifrom motion.

1.4. Relative motion :

Equation (2) and (3) can be written as

$$\ddot{\bar{r}}_1 = G\frac{m_2}{r^3}\,\bar{r} \rightarrow (6)$$

and

$$\ddot{\bar{r}}_2 = -G\frac{m_1}{r^3}\,\bar{r} \rightarrow (7)$$

subtructing (6) from (7), we get

$$\ddot{\bar{r}}_2 - \ddot{\bar{r}}_1 = -G(m_1 + m_2)\frac{\bar{r}}{r^3}$$

$$\Rightarrow \ddot{\bar{r}} = \frac{G(m_1 + m_2)}{r^2}\left(-\frac{\bar{r}}{r}\right)$$

$$\Rightarrow \ddot{\bar{r}}\ \frac{\mu}{r^2}\left(-\frac{\bar{r}}{r}\right) \rightarrow (8)$$

where $\mu = G(m_1 + m_2)$

and $\bar{r}_1 + \bar{r} = \bar{r}_2$

$$\therefore \quad \bar{r} = \bar{r}_2 - \bar{r}_1$$

Which gives motion of the mass m_2 relative to the mass m_1. Clearly this shows that planet (or satellite) is attracted towards the sun (earth) and the law of motion is governed by the inverse square law $\left(\equiv \frac{M}{r^2}\right)$ and is directed towards the centre of attruction.

In fig-1, if the mass m_1 at S is taken as the sun, m_2 at P is taken as the planet then, the motion of a planet is also governed by the inverse square and is direted towards the sun. This principle is also applicable in case of the motion of artificial satellite with respect to the earth as the outcome of two-body problem.

The equation of motion being a 2nd order vector differntial equation must have got two arbitrary vector constants into general solution. But the two vector constants are equevelent to 6 scalar constants. Here for complete determination of the motion of the planet (or satellite). We must know six conditions.

It we know the position (x, y, z) and velocity $(\dot{x}, \dot{y}, \dot{z})$ of the body at one instant we can know the motion (in Newtonian sense) completely. Otherwise the known position of the body at phase-space can help us in knowing the motion under the inverse osquare law (8).

Now taking cross product of (8) with $\bar{r}$, we get

$$\bar{r} \times \ddot{\bar{r}} = 0$$

$$\Rightarrow \bar{r} \times \ddot{\bar{r}} = \text{Constant} = \bar{h} \text{ ---- } (9)$$

Where $\bar{h}$ is the constant angular momentum vector.

Now if we take $\bar{r} = \hat{i}\xi + \hat{j}\eta + \hat{k}\xi$ with respect to sun as orgin. This from (9) we can get.

$$\bar{r} \times \dot{\bar{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \xi & \eta & \xi \\ \dot{\xi} & \dot{\eta} & \dot{\xi} \end{vmatrix} = \bar{h} = \hat{i}A + \hat{j}B + \hat{k}C$$

i components $\eta\dot{\xi} - \dot{\eta}\xi = A$

j componets $\dot{\xi}\varsigma - \xi\dot{\varsigma} = B$

k componets $\xi\dot{\eta} - \eta\dot{\xi} = C$

multiplying by ξ, η and ς respectively and then adding we get

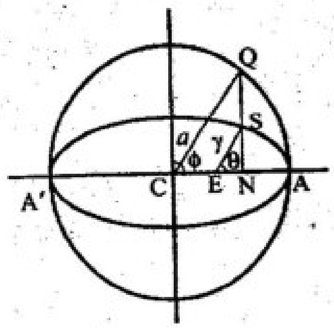
$A\xi + B\eta + C\varsigma = 0$

This is the equation of a plane passing though the orgin. (Here the sun)

Hence the motion of every planet or salellite takes place in a plane passing through the orgin (i.e. sun) otherwise motion of planet is a plane orbit.

1.5. Kepler's Equation :

Let m_1 be the mass of the satellite and m_2 be the mass of the earth (or sun) at E. at the fows of the elliptic orbit with respect to E as pole and EA(CA) as initial line. Let the polar co-ordinates of S be (r,Q) where Q is measured from EA. A being the position of perige (perihelion)



The angle $\angle SEA = Q$ is called the true anomaly. Draw the auxiliary circle with the centre C of the ellipse as its centre and the semi-major axis 'a' and semi-minor axis $b = a\sqrt{1-e^2}$ SN is drawn perpendicular to AA' and SN is produced to meet the auxiliary at Q and $\angle QAC = \phi$ is called ecentic anomaly of the planet or satillite.

The rate of description of the total angle 2π in time T.,i.e. time for one complete revolution by the elliptic orbit by the sutellite (or planet) is called the mean angular motion (or speed) n.

$$n = \frac{2\pi}{T}$$

The mean anomaly 'm' is defined as

$$m = n(t - \tau)$$

Where τ is the time of passage of the price A by the satellite.

$$\therefore m = \frac{2\pi}{T}(t - \tau)$$

Now $r\cos\theta = EN$

$$= CN - CE$$

$$= a\cos\phi - ae$$

Where 'e' is the eccenticity

$$= a(\cos\phi - e) \text{ ----- (i)}$$

$$r\sin\theta = SN$$

$$= \frac{b}{a} QN$$

$$= \frac{b}{a} a\sin\phi$$

$$= b\sin\phi$$

$$= a\sqrt{1-e^2}\sin\phi \text{ ----- (ii)}$$

solving (i) and (ii), we get

$$r^2 = a^2(\cos\phi - e)^2 + a^2(1-e^2)\sin^2\phi$$

$$\Rightarrow r^2 = a^2\left[\cos^2\phi - 2\cos\phi\, e + e^2 + \sin^2\phi - e^2\sin^2\phi\right]$$

$$\Rightarrow r^2 = a^2\left[1 - 2e\cos\phi + e^2\cos^2\phi\right]$$

$$\Rightarrow r^2 = a^2(1 - e\cos\phi)^2$$

$$\Rightarrow r = a(1 - e\cos\phi) \text{ ------------- (iii)}$$

Again from the eq^n of the elliptic or bit

$$\Rightarrow \frac{\ell}{\gamma} = 1 + e\sin\theta, \text{ where } l = \frac{b^2}{a} = \frac{a^2(l-e^2)}{a} = \frac{h^2}{\mu} = a(l-e^2)$$

Differention with respect to time

$$\Rightarrow -\frac{\ell}{r^2}\dot{r} = -e.\ \sin\theta\,\dot{\theta}$$

$$\Rightarrow \dot{\gamma} = e\sin\theta\,.\,\dot{\theta}\,\frac{\gamma^2}{a(1-e^2)}$$

$$= \frac{e\sin\theta}{a(1-e^2)} h$$

$$= \frac{e\sin\theta}{a(1-e^2)}\sqrt{\mu a(1-e^2)}$$

$$= \frac{e\sin\theta\sqrt{\mu}}{\sqrt{a(1-e^2)}}$$

But from (3) $\dot{r} = a\dot{e}\sin\phi \,.\, \dot{\phi}$

$$\Rightarrow a\,\dot{e}\,\sin\phi.\ \dot{\phi} = \sqrt{\frac{\mu}{a(1-e^2)}}\ \dot{e}\sin\phi$$

$$\Rightarrow \dot{\phi} = \sqrt{\frac{\mu}{a^3(1-e^2)}}\ \frac{\sin\theta}{\sin\phi}$$

$$= \sqrt{\frac{\mu}{a}}\ \frac{\sin\theta}{a\sqrt{1-e^2}\sin\phi}$$

$$= \sqrt{\frac{\mu}{a}}\ \frac{1}{\gamma}$$

or $a(1 - e\cos\phi)\,d\phi = \sqrt{\frac{\mu}{a}}\,dt$ | using eq^n (ii) | using eq^n (iii)

Integrating above eq^n we get

$$a(\phi - e\sin\phi) = \sqrt{\frac{\mu}{a}}(t - \tau)$$

where $\phi = 0$ at $t = \tau$, the time of crossing the perihelon (or perige)

$$\therefore\ \phi - e\sin\phi = \sqrt{\frac{\mu}{a^3}}(t - \tau) \quad \text{------ (iv)}$$

Since the rate of description of the total area $= \pi ab$ in period T

$$\therefore\ T = \frac{\pi ab}{T} = \frac{\pi a.a\sqrt{1-e^2}}{T}$$

$$= \frac{\pi a^2\sqrt{1-e^2}}{T}$$

But it is equivalent to

$$\frac{dA}{dt} = \frac{1}{2}h = \frac{1}{2}r^2\dot{\theta}$$

$$\Rightarrow \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}\sqrt{\mu a(1-e^2)}$$

$$\therefore\ \frac{\pi a^2\sqrt{1-e^2}}{T} = \frac{1}{2}\sqrt{\mu a(1-e^2)}$$

$$\text{or}\ \frac{2\pi}{T}a^2 = \sqrt{\mu a}$$

$$\text{or}\ \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

$$\text{or } n = \sqrt{\frac{\mu}{a^3}}$$

$$\text{or } n^2a^3 = \mu.$$

Hence from (iv)

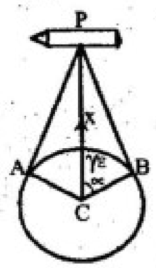
$$\phi - e\sin\phi = n(t - \tau)$$

$$\therefore \phi - e\sin\phi = m \qquad \left| \; \therefore n = \sqrt{\frac{\mu}{a^3}} \text{ and } m = n(t - \tau) \right.$$

which is the Kepler's equation.

Example 1. : Communication satillite of the earth are placed in circular orbit on the equatarid plane so that the remain stationary with respect to the surface of the earth. Determine the minimum number of satillite required for every place and the equation to be in view of at rest one satellite.

Solution : Communication satellites are gio-stationary satellite in the that they remain stationary with respect to the surface of the earth. Therefore a sotellite needs to describe the whole equatarial plane in 1 day = 24 hours. i.e. to describe 360^0 angle at the centre. It r_E is the radius of the earth, x is the distance of the satellite P from the earth's surface and α is the angle made at the centre C (as in fig.)



then

$$\cos\alpha = \frac{r_E}{r_E + x} \quad \text{--------} (1)$$

since

$$r_E = 6378 \text{ km}$$

$$\text{But } T^2 = \frac{4\pi^2}{\mu} a^3 \quad \text{-------------} (2)$$

and $a = (r_E + x)$

Also T = 1 day = 24 hours

$$= 24 \times 60 \times 60 \text{ sec}$$

$$= 8.64 \times 10^4 \text{ sec}$$

$\therefore$ From (2)

$$\left(8.64 \times 10^4\right)^2 = \frac{4\times(3.14115)^2}{398603.6}(r_E + x)^3$$

since GM = M = 398603.6 km³/sec

$$\Rightarrow r_E + x = \frac{(74.6496 \times 398603.6)^{\frac{1}{3}}}{(39.46729329)^{\frac{1}{3}}}$$

$$\Rightarrow r_E + x = \frac{(8.64 \times 10)^{\frac{2}{3}} \times 10^2 \times 73.594790}{(39.476)^{\frac{1}{3}}}$$

$$\Rightarrow r_E + x \cong 42241$$

$$\therefore \text{ From (1) } \cos\alpha = \frac{6378}{42241}$$

$$= 0.15099$$

$$\therefore \infty = 81^0.31$$

$$\therefore 2\infty = 1620$$

$\therefore$ To cover 360°, we require at least 3 satellites.

1.6. Determination of Orbit :

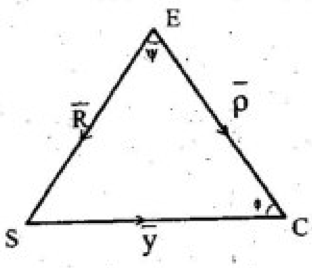
The geocentric position of body (like satellite) means the determination of the required six elements in a celestial sphere by means of observation. Otherwise with the information of the six elements, it is possible to determine the position of the body. But it is essential to take into account of approximation the method of observation. Of course the orbit result is affected by the motion of the earth which is not case in helicuntric system of referrence where the sum (or other stars) as the centre of reference.

1.6.1 Laplace method of determination of orbit :

Let S, E and C be the position of the sun, the earth and the comet (or other body) respectively

where

$\overline{ES} = \overline{R}$, $\overline{SC} = \overline{r}$, $\overline{EC} = \overline{\rho}$



so that

$$r^2 = \rho^2 + R^2 - 2PR\cos\tau \quad \text{------ (1)}$$

$$\bar{R} + \bar{r} = \bar{\rho} = \rho\,\hat{\rho}' \quad \text{------- (2)}$$

$$\bar{R}' + \bar{r}' = \rho'\,\hat{\rho} + \rho\,\hat{\rho}' \quad \text{------- (3)}$$

$$\text{and } \bar{R}'' + \bar{r}'' = \rho''\,\hat{\rho} + 2\rho'\,\hat{\rho}' + \rho\,\hat{\rho}'' \quad \text{------- (4)}$$

where dash stands for differentation with respect to a newly defined time But, from the eq^n motion

$$\bar{r}'' = -\frac{M}{r^3}\bar{r}$$

$$= -\frac{\bar{r}}{r^3}$$

$$\bar{R}'' = -\frac{M}{R^3}\bar{R} = -\frac{\bar{R}}{R^3}$$

where subject to the newtime τ, μ is choosen as unity

$\therefore$ (4) can be transformed to

$$-\frac{\bar{R}}{R^3} - \frac{\bar{r}}{r^3} = \rho''\,\hat{\rho} + 2\rho'\,\hat{\rho}' + \rho\,\hat{\rho}''$$

$$\Rightarrow -\frac{\bar{R}}{R^3} - \frac{\left(\bar{P}-\bar{R}\right)}{r^3} = \rho''\,\hat{\rho} + 2\rho'\,\hat{\rho}' + \rho\hat{\rho}'' \quad \text{using (1)}$$

$$\Rightarrow -\frac{\bar{P}}{r^3} - \left(\frac{1}{r^3} - \frac{1}{R^3}\right)\bar{R} = \rho''\rho + 2\rho'\,\hat{\rho}' + \rho\hat{\rho}'' \quad \text{------ (5)}$$

As mentioned above, the new modified time τ is defined as $\tau = R(t - t_0)$.
where t_0 is the intant of consideration.

The unit of mass is taken as the mass of the sun and the unit of time in such a way so that $k = 1$. Hence the prime denotes differentiation with respect to the newly define time τ.
Laplace methods deals with the determination of $\hat{\rho}'$ and $\hat{\rho}''$ for determination of the orbit by means of the necessary three observation.

We shall choose $\hat{\rho}$ at some instant say τ_0 so that it can be written means of Taylor's series as

$$(\hat{\rho})_\tau = (\hat{\rho})_{\tau=0} + \tau(\hat{\rho}')_{\tau=0}\ \frac{\tau^2}{\angle 2}(\hat{\rho}'')_{\tau=0} + \ldots\ldots \quad \text{------- (6)}$$

The observed values cannot be taken as granted as accurate value, therefore they will be same amount of error. But if we choose the time of middle as the first instant (τ_0) of observation then the expected error can be minimised in great extent.

We assume that $(\hat{\rho}')_o$, one $(\hat{\rho}'')_o$ are known. Therefore $\hat{\rho}'$ and $\hat{\rho}''$ can be assumed with known values of $(\hat{\rho}')_o$ and $(\hat{\rho}'')_o$.

Taking dot product of (5) with $(\hat{\rho} \times \hat{\rho})$ and $(\hat{\rho} \times \hat{\rho}'')$ successively we get

$$\left(\frac{1}{r^3}-\frac{1}{R^3}\right)\left[\hat{\rho}, \hat{\rho}' \bar{R}\right] = \rho\left[\hat{\rho}, \hat{\rho}' \hat{\rho}''\right] \quad \text{------} (7)$$

and

$$\left(\frac{1}{r^3}-\frac{1}{R^3}\right)\left[\hat{\rho}, \rho'', \bar{R}\right] = \rho'\left[\hat{\rho}, \hat{\rho}' \hat{\rho}''\right] \quad \text{------} (8)$$

Since $\hat{\rho}$, ρ'', $\bar{R}$ are known, therefore we can solve for ρ and r from (5) and (7) with these value of r, we can get ρ' from (8)

Hence from (3) we can get $\bar{r}'$. Thus with the known values of $\bar{r}$ (position) and $\bar{r}'$ (velocity) the orbit elements can completely determined.

1.6.2 Gauss's method to dertmine the elements of orbit :

In Laplace method, we dertimine the value of $\hat{\rho}'$ and $\hat{\rho}''$ ($\hat{\rho}$ being the direction of the position of the body or satellite) to know the position $\bar{r}$ and velocity $\bar{r}'$ expanding $\hat{\rho}$ by Taylor's series

$$(\hat{\rho}) = (\hat{\rho})_0 + \tau(\hat{\rho}')_0 \frac{\tau^2}{\angle 2}(\hat{\rho}'')_0 + \ldots\ldots$$

But in Gausses method, we truncate the dynamical theorical results of f and g series defing time suitably given by

$$\bar{r} = \bar{r_o} + \bar{r_o}' + \frac{t^2}{\angle 2}\bar{r_o}'' + \frac{t^3}{\angle 3}\bar{r_o}''' + \ldots\ldots$$

$\bar{r_o}$ = stands for position vector at $t = t_o$

$$\bar{r} = f\,\bar{r_o} + g\,\bar{r_o}' \quad \text{-----------} (1)$$

$$\text{where } f = 1 - \frac{1}{2}\sigma t^2 + \frac{1}{2}\sigma \tau t^3 + \ldots\ldots$$

$$g = t - \frac{1}{6}\sigma t^3 + \frac{1}{4}\tau t^4 + \ldots\ldots$$

$$\sigma = \frac{1}{r_o^{\,3}} = \frac{r_o . r_o}{r_o^{\,5}}$$

$$\tau = \frac{\bar{r}_o . \bar{r}_o'}{r_o^{\,2}} = \frac{\bar{r}_o'}{r_o}$$

Of course the time of superation of the position vector for three observations is to taken small

Since the orbit is a planeorbit and if

$$\bar{r}_2 = c_1\,\bar{r}_2 + c_3\,\bar{r}_3 \qquad \text{------------ (2)}$$

Taking cross product of (2) with $\bar{r}_1$ and $\bar{r}_3$ successively we get

$$\bar{r}_1 \times \bar{r}_2 = c_3\,\bar{r}_3 \times \bar{r}_2 \text{ and } \bar{r}_2 \times \bar{r}_3 = c_1\,\bar{r}_1 \times \bar{r}_3$$

$$\therefore \left(r_1.r_2\right) = c_3\left(r_1.r_3\right),\ \left(r_2.r_3\right) = c_1\left(r_1.r_3\right)$$

where $[r_i, r_j]$ represents the area of the triangle with r_i, r_j as two adjacentsides

$$\therefore c_1\ \frac{[r_2.r_3]}{[r_1.r_3]},\quad c_3 = \frac{[r_1.r_2]}{[r_1.r_3]} \qquad \text{------(3)}$$

But the three equation's of is given by (2) are not linearly independent. But if we make use of $\bar{r} = \bar{\rho} - \bar{R}$ then we can write (2) as

$$\bar{\rho}_2 - \bar{R}_2 = C_1\left(\bar{\rho}_1 - \bar{R}_1\right) + C_3\left(\bar{\rho}_3 - \bar{R}_3\right)$$

$$\Rightarrow C_1\,\rho_1 - \rho_2 + C_3\rho_3 = C_1\,R_1 - R_2 + C_3\bar{R}_3 \qquad \text{------- (4)}$$

which show's the ρ's are linearly independent and $\bar{R}_i$ $(i = 1, 2, 3, \ldots)$ are the distance of the sun from the observer instead of the centre of the earth to avoid effect of parallax.

Now are take

$$T_1 = k\left(t_3 + t_2\right),\ T_2 = k\left(t_3 + t_1\right)$$

$$T_3 = k\left(t_2 + t_1\right)$$

so that $T_1 = \tau_3,\ \tau_2 = \tau_3 - \tau_2,\ T_3 = -\tau_1$
since we modified time as

$$\tau = k\left(t - t_o\right)$$

where to stands for the time t_2 (time of mid observer) that corresponds $\bar{r}_2$

In the light of (1) we can write

$$\bar{r}_1 = f_1 \bar{r}_2 + g_1 \bar{r}_2' \text{ ----- (5)}, \quad r_3 = f_3 \bar{r}_2 + g_3 \bar{r}_2' \text{ ----------- (6)}$$

Now making use of fand g series (1) we can write (1)

$$\left.\begin{aligned} f_1 &= 1 - \frac{1}{2}\sigma T_3^2, \quad g_1 = T_3\left(1 - \frac{1}{6}\sigma T_3^2\right) \\ f_3 &= 1 - \frac{1}{2}\sigma T_1^2; \quad g_3 = T_1\left(1 - \frac{1}{6}\sigma T_1^2\right) \end{aligned}\right\} \text{ ----- (7)}$$

Taking the cross product (5) with $\bar{r}_1$ and $\bar{r}_2$ successively

$$o = f_1 \bar{r}_1 \times \bar{r}_2 + g_2 \bar{r}_1 \times \bar{r}_2',$$

$$\Rightarrow \bar{r}_1 \times \bar{r}_2 = -\frac{f_1}{g_2} \bar{r}_1 \times \bar{r}_2$$

$$\text{and } \bar{r}_1 \times \bar{r}_2 = o - g_1 \bar{r}_2' \times \bar{r}_2$$

$$= -g_1 \bar{r}_2 \times \bar{r}_2'$$

$$\Rightarrow -\frac{1}{g_1} \bar{r}_1 \times \bar{r}_2 = \bar{r}_1 \times \bar{r}_2'$$

Again taking cross product of (5) with $\bar{r}_1 \in \bar{r}_2$ successively we get

$$\bar{r}_1 \times \bar{r}_3 = \frac{f_1 g_3 - f_3 g_1}{-g_1} \bar{r}_1 \times \bar{r}_2$$

$$\therefore \frac{[r_1.r_3]}{f_1 g_3 - f_3 g_1} = \frac{[r_1.r_2]}{-g_1} = \frac{[r_2.r_3]}{g_3}$$

$$\text{and } \bar{r}_2 \times \bar{r}_3 = \frac{g_3}{-g_1} \bar{r}_1 \times \bar{r}_2$$

$$\text{or } \frac{\bar{r}_2 \times \bar{r}_3}{g_3} = \frac{1}{-g_1} \bar{r}_1 \times \bar{r}_2$$

from (3)

$$C_1 = \frac{T_1\left(1 - \frac{1}{6}\sigma T_3^2\right)}{f_1 g_3 - f_3 g_1}$$

$$C_3 = \frac{T_3\left(1 - \frac{1}{6}\sigma T_1^2\right)}{f_1 g_3 - f_3 g_1}$$

But

$$f_1 g_3 - f_3 g_1 = (T_1 + T_3)[1 - \frac{1}{6}\sigma(T_1 + T_3)^2]$$
$$= T_2 [1 - \frac{1}{6}\sigma T_2^2]$$

$\because T_1 - T_2 = T_3$

$$\therefore C_1 = \frac{T_1}{T_2}[1 - \frac{1}{6}\sigma(T_2^2 - T_1^2)]$$
$$C_3 = \frac{T_3}{T_2}[1 - \frac{1}{6}\sigma(T_2^2 - T_3^2)]$$

Taking det product (4) with $\hat{\rho}_1 \times \hat{\rho}_3$

$$\Rightarrow -\hat{\rho}_2(\hat{\rho}_1 \times \hat{\rho}_3) = (\hat{\rho}_1 \times \hat{\rho}_3)\left[C_1\overline{R_1} - \overline{R_2} + C_3\overline{R_3}\right]$$
$$\Rightarrow (\hat{\rho}_1 \times \hat{\rho}_3) \cdot \left[\overline{\rho_2} + C_1\overline{R_1} - \overline{R_2} + C_3\overline{R_3}\right] = 0$$

From this eq^n we can get

$$\rho_2 + \frac{A}{r_2^3} = 0 \quad \left|\because c_1,\ c_3 \text{ are function of } \sigma = \frac{1}{r_2^3}\right.$$

But $r_2^2 = P_2^2 + R_2^2 - 2P_2R_2 \cos \tau$.

$\therefore$ we can solve for ρ_2 and r_2 from these two equations. Similarly we can solve for ρ_1 and ρ_3 and r_1, r_2, r_3 can also be determined from $\bar{r} = \bar{\rho} - \bar{R}$

Hence, $\overline{r_2}$ and $\overline{r_2}'$ can be determined from $\overline{r_1} = f_1\overline{r_2} + g_1\overline{r_2}'$

$\overline{r_3} = f_3\overline{r_2} + g_3\overline{r_2}'$

Hence the elements of the orbit can be known for the known values of $\overline{r_2}$ and $\overline{r_2}'$.

Exercise

Q.1. Show that the motion of a planet relative to the sun is govern by the inverse square law.

Q.2. Show that the nature of orbit of the planet of satellite depends on the velocity of projection.

Q.3. Express of eccentric anomaly ϕ in terms of series of e (eccentricity) and m (mean anomaly)

Q.4. An artificial satellite is released at an altitude of 400 k.m. with speed 8.85 km/see horizontally. Find the perigce, apogce and the time period. Also determine the total energy per unit mass assuming the centre of the earth as the centre of force, radius of the earth as 6378 km and GM = 398603.6 km/sec^2.

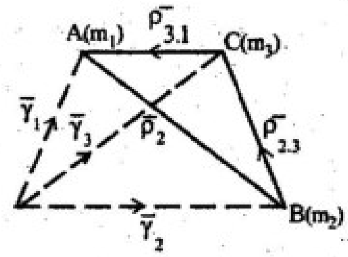
Three body problem :

2.1. Defination :

There bodies attracted each other according to Newton's law of gravitation. To determine the motion of the bodies which can move freely in space in any manner intially. These body is known as the three body problem. Sun, moon and earth are the example of three body problem.

2.2. Motion of the centre of mass :

Let m_1, m_2, m_3 be three masses with $\bar{r}_1, \bar{r}_2, \bar{r}_3$ respect to the find orgin O. Let A, B, C be the position occupied by the masses m_1, m_2, m_3 respectively.



Sppose that

$$\overline{AB} = \overline{\rho_{12}} = \bar{r}_2 - \bar{r}_1$$
$$\overline{BC} = \overline{\rho_{23}} = \bar{r}_3 - \bar{r}_2$$
$$\overline{CA} = \overline{\rho_{31}} = \bar{r}_1 - \bar{r}_3$$

The equations of the motion of the masses are

$$m_1 \ddot{\bar{r}}_1 = \frac{k^2 m_1 m_2}{\rho_{12}^2} \cdot \frac{\bar{\rho}_2}{\rho_{12}} + \frac{k^2 m_1 m_3}{\rho_{31}^2} \cdot \frac{\bar{\rho}_{12}}{\rho_{13}}$$

$$= K^2 \left[\frac{m_1 m_2}{\rho_{12}^3} \rho_{12} - \frac{m_3 m_1}{\rho_{31}^3} \rho_{31} \right] \quad \text{------- (1)}$$

Similarly

$$m_2 \ddot{\bar{r}}_2 = k^2 \left[\frac{m_2 m_3}{\rho_{23}^2} \bar{\rho}_{23} - \frac{m_1 m_2}{\rho_{12}^3} \bar{\rho}_{12} \right] \quad \text{------- (2)}$$

$$m_3 \ddot{\bar{r}}_3 = k^2 \left[\frac{m_3 m_1}{\rho_{31}^3} \bar{\rho}_{31} - \frac{m_2 m_3}{\rho_{23}^3} \bar{\rho}_{23} \right] \quad \text{------- (3)}$$

(1) + (2) + (3) $\Rightarrow$

$$m_1 \ddot{\bar{r}}_1 + m_2 \ddot{\bar{r}}_2 + m_3 \ddot{\bar{r}}_3 = k^2 \left[\bar{0}\right] = \bar{0}$$

$$\Rightarrow m_1\dot{\bar{r_1}} + m_2\dot{\bar{r_2}} + m_3\dot{\bar{r_3}} = \overline{C_1}$$

$$\Rightarrow m_1\overline{r_1} + m_2\overline{r_2} + m_3\overline{r_3} = \overline{C_1}t + C_2 \text{-----}(4)$$

Where $\overline{C_1}$ & $\overline{C_2}$ are two arbitrary constant vector. Let $\overline{R}$ be the position vector of the centre of mass of the saytem consisting of the mass m_1, m_2, m_3.

$$\therefore \overline{R} = \frac{m_1\overline{r_1} + m_2\overline{r_2} + m_3\overline{r_3}}{m_1 + m_2 + m_3}$$

$$\therefore \overline{R} = \frac{m_1\overline{r_1} + m_2\overline{r_2} + m_3\overline{r_3}}{M} \text{------}(5)$$

Where $M = m_1 + m_2 + m_3$

= total mass of the system.

$$(5) \Rightarrow m_1\overline{r_1} + m_2\overline{r_2} + m_3\overline{r_3} = M\overline{R} \text{-----}(6)$$

$$\therefore (4) \Rightarrow M\overline{R} = \overline{C_1}t + \overline{C_2}$$

$$\Rightarrow \overline{R} = \frac{\overline{C_1}}{M}t + \frac{\overline{C_2}}{M}$$

$$= at + b \text{ ----- (7) where } \bar{a} = \frac{\overline{C_1}}{M}, \bar{b} = \frac{\overline{C_2}}{M}$$

(7) Shows that the centre of mass of the three masses is either at rest or moves in a straight line with constant velocity.

The eq^n (7) is also known as the first integral of three body problem.

Here the vector

$\bar{a} = (a_1, a_2, a_3)$

$\bar{b} = (b_1, b_2, b_3)$

involve six scalar constant

2.3. Integral of Energy :

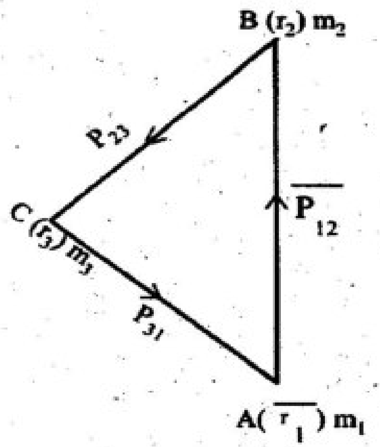
The eq^n of motion three masses m_1, m_2, m_3 place at the points $A(\overline{r_1})$, $B(\overline{r_2})$, $C(\overline{r_3})$ are deducted

$$m_1\ddot{\bar{r_1}} = k^2\left[\frac{m_1m_2}{\rho_{12}^3}\bar{\rho}_{12} - \frac{m_3m_1}{\rho_{31}^3}\bar{\rho}_{31}\right] \text{-------}(1)$$

$$m_2\ddot{\bar{r_2}} = k^2\left[\frac{m_2m_3}{\rho_{23}^2}\bar{\rho}_{23} - \frac{m_1m_2}{\rho_{12}^3}\bar{\rho}_{12}\right] \text{-------}(2)$$

$$m_3 \ddot{\bar{r}}_3 = k^2\left[\frac{m_3 m_1}{\rho_{31}^3}\bar{\rho}_{31} - \frac{m_2 m_3}{\rho_{23}^3}\bar{\rho}_{23}\right] \quad \text{-------} (3)$$

By taking the dot product of the eqⁿ (1) by $\ddot{\bar{r}}_1$ we get



$$m_1 \dot{\bar{r}}_1 \cdot \ddot{\bar{r}}_1 = k^2\left[\frac{m_1 m_2}{\rho_{12}^3}\dot{\bar{r}}_1\cdot\bar{\rho}_{12} - \frac{m_3 m_1}{\rho_{31}^3}\dot{\bar{r}}_1\cdot\bar{\rho}_{31}\right] \rightarrow (1)'$$

$$m_2 \dot{\bar{r}}_2 \cdot \ddot{\bar{r}}_2 = k^2\left[\frac{m_2 m_3}{\rho_{23}^3}\dot{\bar{r}}_2\bar{\rho}_{23} - \frac{m_1 m_2}{\rho_{12}^3}\dot{\bar{r}}_2\cdot\bar{\rho}_{12}\right] \rightarrow (2)'$$

$$m_3 \dot{\bar{r}}_3 \cdot \ddot{\bar{r}}_3 = k^2\left[\frac{m_3 m_1}{\rho_{31}^3}\dot{\bar{r}}_3\cdot\bar{\rho}_{31} - \frac{m_2 m_3}{\rho_{23}^3}\dot{\bar{r}}_3\cdot\bar{\rho}_{23}\right] \rightarrow (3)'$$

Now (1)′ + (2)′ + (3)′ ⇒

$$\sum_{\alpha=1}^{3} m_\alpha \dot{\bar{r}}_\alpha \cdot \ddot{\bar{r}}_\alpha = k^2\left[\frac{m_1 m_2}{\rho_{12}^3}\left(\dot{\bar{r}}_1 - \dot{\bar{r}}_1\right)\bar{\rho}_{12} - \frac{m_2 m_3}{\rho_{23}^3}\left(\dot{\bar{r}}_2 - \dot{\bar{r}}_3\right)\bar{\rho}_{23} + \frac{m_3 m_1}{\rho_{31}^3}\left(\dot{\bar{r}}_3 - \dot{\bar{r}}_1\right)\bar{\rho}_{31}\right]$$

$$= k^2\left[\frac{m_1 m_2}{\rho_{12}^2}\hat{\rho}_{12} - \frac{m_2 m_3}{P_{23}^2}\hat{\rho}_{23} + \frac{m_3 m_1}{\rho_{31}^2}\hat{\rho}_{31}\right] \quad \text{-------} (4)$$

Now

$$\frac{d}{dt}\dot{\bar{r}}_\alpha^{\,2}\cdot\ddot{\bar{r}}_\alpha = \frac{d}{dt}\left(\dot{\bar{r}}_\alpha\cdot\dot{\bar{r}}_\alpha\right)$$

$$= 2\dot{\bar{r}}_\alpha\cdot\ddot{\bar{r}}_\alpha$$

$$\Rightarrow \dot{\bar{r}}_\alpha\cdot\ddot{\bar{r}}_\alpha = \frac{1}{2}\frac{d}{dt}\dot{\bar{r}}_\alpha^{\,2}$$

$$= \frac{d}{dt}\dot{\bar{r}}_\alpha^{\,2} \quad \text{-------} (5)$$

and $\quad \frac{d}{dt}\frac{1}{u} = (-1)\,u^{-2}\frac{du}{dt}$

$$= -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\dot{u}$$

$$\frac{\dot{u}}{u^2} = \frac{d}{dt}\frac{1}{u} \text{ ------ (6)}$$

Using (5) & (6) in (4) we get

$$\sum_{\alpha=1}^{3} m_\alpha \frac{1}{2}\frac{d}{dt}\dot{r}_\alpha^{\,2} = -k^2\left[-m_1m_3\frac{d}{dt}\left(\frac{1}{\rho_{12}}\right) - m_2m_3\frac{d}{dt}\left(\frac{1}{\rho_{23}}\right) - m_3m_1\frac{d}{dt}\left(\frac{1}{\rho_{31}}\right)\right]$$

$$\Rightarrow \frac{d}{dt}\left(\sum_{\alpha=1}^{3} m_\alpha \frac{1}{2}\dot{r}_\alpha^{\,2}\right) = k^2\frac{d}{dt}\left(\frac{m_1m_3}{\rho_{12}} + \frac{m_2m_3}{\rho_{23}} + \frac{m_3m_1}{\rho_{31}}\right)$$

$$\Rightarrow \frac{d}{dt}T = -\frac{d}{dt}(V)$$

$$\Rightarrow \frac{dT}{dt} = -\frac{dV}{dt} \text{ ------------ (7)}$$

where $T = \sum_{\alpha=1}^{3}\frac{1}{2}m_\alpha \dot{r}_{\alpha o}$

= Kinetic energy consisting of the massis m_1, m_2 & m_3 and

$$V = -\frac{k^2m_1m_3}{\rho_{12}} + \frac{k^2m_2m_3}{\rho_{23}} + \frac{k^2m_3m_1}{\rho_{31}}$$

= Potential energy of the system.

(7) can be written as

$$\frac{d}{dt}(T + V) . 0$$

$$\Rightarrow T + V = E = a \text{ constant ----------- (8)}$$

This equation (8) is called the integral of energy. This equation says that the sum of the K.E.& P.E. of a system consisting of three masses is conserved. In (8) the arbitrary scalar constant E is called the total energy of the system.

Summary : The integrals of (1), (2) & (3) are

$$\bar{R} = \bar{a}t + \bar{b} \text{ ------------- (A)}$$

$$\sum_{r=1}^{3} r_\alpha \times m_\alpha \overline{r_\alpha} = \overline{b} \quad \text{----------- (b)}$$

$$T + V = E \quad \text{--------------- (C)}$$

with usual manings of the symbols. The total order of the equations of motion of the system is

$2 \times 3 + 2 \times 3 + 2 \times 3$

$= 18$

Therefore the general salutions of the equations (1), (2) & (3) must contain (18) arbitrary constant

The sol^n (A) contains 3 + 3 = 6 arbitrary scalar constants.

The sol^n (B) contains 3 arbitrary scalar constants.

The sol^n (C) contains 1 arbitrary scalar constants.

Therefore the total number of arbitrary scalar constans involves in the integral's (A), (B), (C) = 6 + 3 + 1 = 10 which is the less then 18.

Therefore the solutions D, E, F cannot be the general solution of the equations of motion of the system consisting of three bodies. Hence the problem remains unsolvable ofcourse some solutions under certain restrictions can be found in 1887.

H. brunt demonstrated that the solutions are independent. Also in 1872 another astronomor poincare also found the some which was demonestrated by H. brunt.

2.4. Stationary Solutions of the three body problem :

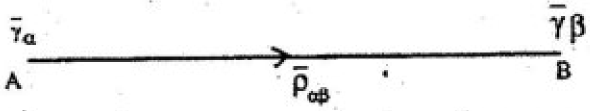
In 1772, Lagrange descover to special salution of the 3-body problem which may be termed as stationary solutions. The stationary solutions of three body problem means one in which the gemetrical configuration of the bodies remain unchanged assuming then to be projected in a plane initially.

The invariance of the geometrical configuraction can take place in to different ways as stated below.

1. It the motion of the masses is such that their mutual distances from each other remain unchanged through out the motion which can occur if the bodies rotate about there centre of mass as if they are rigidly connected in the same plane throughout the motion.
2. It the mutual distance between each pair of bodies expand or contract in the same ratio. So that shape of the patterns of the points remain unaltered.

2.5 The n-body problem :

The solar system with sum as the dominant mass is a true model of n-bodies with mass $m_\alpha(\alpha = 1, 2, ...n)$; Let $\bar{r}_\alpha$ be the position vector of the mass m_α relative to the point O. Let $\bar{\rho}_{\alpha\beta}$ be the position vector of the mass m_β relative to the mass m_α.



We assume that the measser passes spherical system for which they can be consider as point mass and only external forces are their mutual attraction.

The eq^n of motion of the α^{th} mass is

$$m_\alpha \ddot{\bar{r}}_\alpha = \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{2}} = \frac{\overline{\rho_{\alpha\beta}}}{\rho_{\alpha\beta}}$$

$$\Rightarrow m_\alpha \ddot{\bar{r}}_\alpha = \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{3}} \overline{\rho_{\alpha\beta}} \quad ------(1)$$

$$\text{(1) gives} \quad \sum_{\alpha=1}^{n} m_\alpha \ddot{\bar{r}}_\alpha = \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \sum_{\beta=1}^{n} \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{3}} \overline{\rho_{\alpha\beta}} \quad ------(2)$$

Now

$$\frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{3}} \overline{\rho_{\alpha\beta}} + \frac{k^2 m_\beta m_\alpha}{\rho_{\beta\alpha}^{3}} \overline{\rho_{\beta\alpha}}$$

$$= \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{3}} \overline{\rho_{\alpha\beta}} - \frac{k^2 m_\beta m_\alpha}{\rho_{\beta\alpha}^{3}} \overline{\rho_{\beta\alpha}}$$

$$= 0$$

$$\therefore \text{(2) reduces to} \quad \sum_{\alpha=1}^{n} m_\alpha \ddot{\bar{r}}_\alpha = 0 \quad -------- (3)$$

Let $\bar{R}$ be the position vector of the centre of mass of the system relative to 0

$$\bar{R} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2 + ... + m_n\bar{r}_n}{m_1 + m_2 + ... + m_n}$$

$$= \frac{\sum_{\alpha=1}^{n} m_\alpha \overline{r_\alpha}}{\sum_{\alpha=1}^{n} m_\alpha}$$

$$= \frac{\sum_{\alpha=1}^{n} m_\alpha \overline{r_\alpha}}{M} \quad \text{Where } M = \sum_{\alpha=1}^{n} m_\alpha$$

$$\Rightarrow \sum_{\alpha=1}^{n} m_\alpha \overline{r_\alpha} = M\overline{R}$$

$$\Rightarrow \sum_{\alpha=1}^{n} m_\alpha \ddot{\overline{r_\alpha}} = M\ddot{\overline{R}}$$

$\therefore$ (3) reduces to

$$M\ddot{\overline{R}} = 0$$

$$\Rightarrow \ddot{\overline{R}} = 0$$

$$\Rightarrow \ddot{\overline{R}} = a$$

$$\Rightarrow \overline{R} = \overline{a}t + \overline{b} \quad \text{---------} \ (4)$$

Where $\overline{a}$ & $\overline{b}$ are two arbitrary constant vector from (4) it follows that the locus of thecentre of mass is a straight line.

(4). is known as the first integral of the n-body problem.

Again (1) gives

$$\overline{r_\alpha} \times m\ddot{\overline{r_\alpha}} = \overline{r_\alpha} \times \sum_{\substack{\beta=1 \\ \beta\neq 1}}^{n} \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{\ 3}} \overline{\rho_{\alpha\beta}}$$

$$= \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{3}} \overline{\gamma}_\alpha \times \hat{\rho}_{\alpha\beta}$$

$$\Rightarrow \sum_{\alpha=1}^{n} \bar{\gamma} \times m\ddot{r}_\alpha = \sum_{\alpha=1}^{n} \sum_{\substack{\beta=1 \\ \alpha\neq\beta}}^{n} \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{3}} \overline{\gamma}_\alpha \times \hat{\rho}_{\alpha\beta} \rightarrow (5)$$

Now

$$\frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{\ 3}} \overline{r_\alpha} \times \overline{\rho_{\alpha\beta}} + \frac{k^2 m_\beta m_\alpha}{\rho_{\beta\alpha}^{\ 3}} \overline{r_\beta} \times \overline{\rho_{\beta\alpha}}$$

$$= \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{\ 3}} \left[\overline{r_\alpha} \times \overline{\rho_{\alpha\beta}} - \overline{r_\beta} \times \overline{\rho_{\beta\alpha}}\right]$$

$$= \frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{\ 3}} \left(\overline{r_\beta} - \overline{r_\alpha}\right) \overline{\rho_{\alpha\beta}}$$

$$= -\frac{k^2 m_\alpha m_\beta}{\rho_{\alpha\beta}^{\ 3}} \overline{\rho_{\alpha\beta}} \times \overline{\rho_{\alpha\beta}}$$

$= 0$

$\therefore$ (5) reduces to

$$\sum_{\alpha=1}^{n} \overline{r_\alpha} \times m_\alpha \ddot{\overline{r_\alpha}} = 0 \quad \text{------(6)}$$

let $\overline{\Omega}$ be the angular momentum of the system of masses about 0.

$$\therefore \overline{\Omega} = \sum_{\alpha=1}^{n} \overline{r_\alpha} \times m_\alpha \overline{v_\alpha}$$

$$= \sum_{\alpha=1}^{n} \overline{r_\alpha} \times m_\alpha \dot{\overline{r_\alpha}}$$

$$\therefore \frac{d\overline{\Omega}}{dt} = \sum_{\alpha=1}^{n} \left[\dot{\overline{r_\alpha}} \times m_\alpha \dot{\overline{r_\alpha}} + \overline{r_\alpha} \times m_\alpha \ddot{\overline{r_\alpha}}\right]$$

$$= \sum_{\alpha=1}^{n} \overline{r_\alpha} \times m_\alpha \ddot{\overline{r_\alpha}} \quad \text{--------(7)}$$

$\therefore$ (6) reduces to $\dfrac{d\overline{\Omega}}{dt} = 0$

$\Rightarrow \overline{\Omega}$ = a constant vector $\overline{L}$ (say) ---------- (8)

The equation (8) shows that the total angular monentum of the system about 0 is constant i.e. the total angular monentum of the system about is conserved and equation (8) is also consisting of n bodies. Get is also known as the integral of angullar monentum of n-body problem.

2.6. Equations of relative motion :

Let us consider a system consisting of three bodies with massen m_1, m_2, m_3 placed at the points A, B, C respectively. Let $\overline{r_1}$, $\overline{r_2}$, $\overline{r_3}$ be the position vector of A, B & C respectively relative to a point 0.

$$\text{Let } \overline{\rho_{12}} = \overline{AB} = \overline{r_2} - \overline{r_1}$$

$$\overline{\rho_{23}} = \overline{BC} = \overline{r_3} - \overline{r_2}$$

$$\text{and } \overline{\rho_{31}} = \overline{CA} = \overline{r_1} - \overline{r_3}$$

The equations of the masses are

$$\ddot{\overline{r_1}} = \frac{k^2 m_2}{P_{12}^3} \overline{\rho_{12}} + \frac{k^2 m_3}{P_{13}^3} \overline{\rho_{13}}$$

$$= k^2 \left[\frac{m_2 \overline{\rho_{12}}}{\rho_{12}^3} + \frac{m_3 \overline{\rho_{13}}}{\rho_{31}^3} \right] \text{ --------- (1)}$$

Similarly

$$\ddot{\overline{r_2}} = k^2 \left[\frac{m_3}{\rho_{23}^3} \overline{\rho_{23}} + \frac{m_2}{\rho_{12}^3} \overline{\rho_{12}} \right] \text{ --------- (2)}$$

$$\ddot{\overline{r_3}} = k^2 \left[\frac{m_1}{\rho_{31}^3} \overline{\rho_{31}} - \frac{m_2}{\rho_{23}^3} \overline{\rho_{23}} \right] \text{ --------- (3)}$$

Now (2) – (1) and (3) – (1) ⇒

$$\ddot{\overline{r_2}} - \ddot{\overline{r_1}} = \ddot{\overline{\rho_{12}}}$$

$$= -\frac{k^2}{\rho_{12}^3} (m_1 + m_2)\overline{\rho_{12}} + k^2 m_3 \left(\frac{\overline{\rho_{23}}}{\rho_{23}^3} - \frac{\overline{\rho_1}}{\rho_{13}} \right) \text{ --------- (4)}$$

and

$$\ddot{\overline{r_3}} - \ddot{\overline{r_1}} = \ddot{\overline{\rho_{13}}}$$

$$= -\frac{k^2 (m_1 + m_3)}{\rho_{13}^3} \overline{\rho_{13}} + k^2 m_2 \left(\frac{\overline{\rho_{23}}}{\rho_{23}^3} - \frac{\overline{\rho_{12}}}{\rho_{12}^3} \right) \text{ --------- (5)}$$

(4) is the equation of the mass m_2 relative to the mass m_1 and (5) is the eqⁿ of motion of relative to m_1.

If $m_3 = 0$, it is obvious that the equation

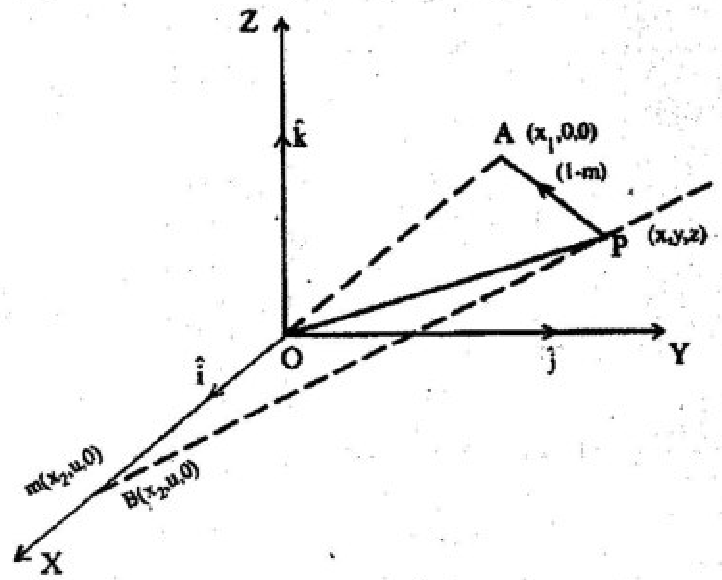
(4) Describes the motion of m_2 around m_1 as in the two body problem.

2.7. The restricted three body problem :

A particular solution of the three body problem results when one of the three masses is so small in comparison to the other two that it can be neglected as far as its gravitional effects one concerned. This may be called an infinitesional body compared with the two finite bodies.

Let the two manssive bodies barring spherical symmetry move about their mass in circular orbits. A third mass, the infinitesional one moves under the combined gravitational attruction of the two but does not influence their motion.

Let m be the mass of the smaller of the two masses and one mass that of the larger. Thus the unit of mass is so choosen that the unit of the time be choosen so that the gravitational constant $k^2{}' = 1$



Let C be the centre of mass 'm' at B and mass 1-m at A. Let us introduce a co-ordinate system C-xyz with C as the orgin so that A and B lie on X -asis. Let P (x, y, z) be the position of the infinitesional mass.

Let $\bar{\upsilon} = \overline{CP}$

$\therefore \bar{\upsilon}$ = velocity of P relative to the moving system C– X Y Z

$= \dot{\bar{r}}\Big]_m$

$\bar{a}$ = acceleration of P relative to the moving system C–X Y Z

$$= \left.\ddot{\bar{r}}\right]_m$$

The equation of motion of the infinitesional motion relative to fixed system at C is

$$= \left.\frac{d\bar{r}^2}{dt^2}\right]_F = -\frac{k^2 m}{BP^2}\cdot\frac{\overline{BP}}{BP} - \frac{k^2(1-m)}{AP^2}\cdot\frac{\overline{AP}}{AP}$$

$$\Rightarrow \left.\frac{d^2\bar{r}}{dt^2}\right]_M + \left.\frac{d\bar{w}}{dt}\times\bar{r}\right]_m + 2w\times\left.\frac{d\bar{r}}{dt}\right]_M + \bar{w}\times\left(\bar{w}\times\bar{r}\right) = -\frac{m\bar{\rho}_2}{\rho_2^{\,3}} - \frac{(1-m)}{\rho_1^{\,3}}\bar{\rho}_1$$

$$\Rightarrow \bar{a} + 2\bar{w}\times\bar{\upsilon} + \bar{w}\left(\bar{w}\times\bar{r}\right) = -\frac{1-m}{\rho_1^{\,3}}\bar{\rho}_1 - \frac{m}{\rho_2^{\,3}}\bar{\rho}_2 \quad \text{--------(1)}$$

Where $\bar{w} = n\hat{k}$, a continious vector relative to n- system.

We have

$\bar{r} = \hat{i}x + \hat{j}y + \hat{k}z$, $\hat{i}, \hat{j}, \hat{k}$ being the unit vectors along $\overline{CX}, \overline{CY}, \overline{CZ}$ respectively

$$\therefore \bar{\upsilon} = \dot{\bar{r}} = \hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}$$

$$\textit{and } \bar{a} = \ddot{\bar{r}} = \hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}$$

$$\bar{w}\times\bar{\upsilon} = n\hat{k}\times\left(\dot{x}\hat{j} + \dot{y}\hat{j} + \dot{z}\hat{k}\right) = n\left(\dot{x}\hat{j} - \dot{y}\hat{i}\right)$$

$$\bar{w}\times\bar{r} = n\hat{k}\times\left(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\right) = nx\hat{j} - ny\hat{i}$$

$$\text{and, } \bar{w}\times\left(\bar{w}\times\bar{r}\right) = n\hat{k}\times\left(nx\hat{j} - ny\hat{j}\right)$$

$$= n^2x\left(-\hat{i}\right) - n^2y\hat{j}$$

$$= -n^2\left(x\hat{i} + y\hat{j}\right)$$

$$\bar{\rho}_1 = \left(x - x_1\right)\hat{i} + y\hat{j} + z\hat{k}$$

$$\bar{\rho}_2 = \left(x - x_2\right)\hat{i} + y\hat{j} + z\hat{k}$$

$\therefore$ The certesian equations of motion are

$$\ddot{x} - 2n\dot{y} = n^2x - \frac{1-m}{\rho_1^{\,3}}\left(x - x_1\right) - \frac{m}{\rho_2^{\,3}}\left(x - x_2\right) \quad \text{--------(2.1)}$$

$$\ddot{y} - 2n\dot{x} = n^2 y - \frac{1-m}{\rho_1^3} y - \frac{m}{\rho_2^3} y \quad \text{--------} (2.2)$$

$$\ddot{z} = -\frac{1-m}{\rho_1^3} z - \frac{m}{\rho_2^3} z \quad \text{--------} (2.3)$$

Now by keplir's third law

$$P = T = \frac{2\pi}{\sqrt{xi}} a^{3/2}$$

$$= \frac{2\pi}{K\sqrt{m_1 + m_2}} a^{3/2}$$

$$\therefore n = \frac{2\pi}{P} = \frac{K\sqrt{m_1 + m_2}}{a^{3/2}} \quad \text{---------} (3)$$

Let us choose the scale of distance so that

$x_2 - x_1 = 1$ i.e. $a = 1$

$\therefore n = 1 \left[\because k = 1, m_1 + m_2 = 1\right]$

Therefore the equations of motion becomes

$$\ddot{x} - 2\dot{y} = x - \frac{1-m}{\rho_1^3}(x - x_1) - \frac{m}{\rho_2^3}(x - x_2) \quad \text{--------} (4.1)$$

$$\ddot{y} + 2\dot{x} = y - \frac{1-m}{\rho_1^3} y - \frac{m}{\rho_2^3} y \quad \text{--------} (4.2)$$

$$\ddot{z} = -\frac{1-m}{\rho_1^3} z - \frac{m}{\rho_2^3} z \quad \text{--------} (4.3)$$

The general problem of determining the motion of the infinitesional mass is, therefore, one requiring six integrals for its complete solution.

Now let us consider a function U defined by

$$U = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-m}{\rho_1} + \frac{m}{\rho_2} \quad \text{--------} (5)$$

where

$$\rho_1^2 = (x - x_1)^2 + y^2 + z^2$$

$$\rho_2^2 = (x - x_2)^2 + y^2 + z^2$$

$$\therefore 2\rho_1 . \frac{\partial \rho_1}{\partial x} = 2(x - x_1)$$

$$2\rho_2 \frac{\partial \rho_2}{\partial x} = 2(x - x_2)$$

Now

$$\frac{\partial \cup}{\partial x} = \frac{1}{2} .2x + (1-m)(-1)\rho_1^{-2} \frac{\partial \rho_1}{\partial x} + m(-1)\rho_2^{-2} \frac{\partial \rho_2}{\partial x}$$

$$= x - \frac{1-m}{\rho_1^2} \frac{\partial \rho_1}{\partial x} - \frac{m}{\rho_2^2} \frac{\partial \rho_2}{\partial x}$$

$$= x - \frac{1-m}{\rho_1^2} . \frac{x - x_1}{\rho_1} - \frac{m}{\rho_2^2} . \frac{x - x_2}{\rho_2}$$

$$= x - \frac{(1-m)(x - x_1)}{\rho_1^3} - \frac{m(x - x_2)}{\rho_2^3}$$

$$\frac{\partial \cup}{\partial y} = y - \frac{1-m}{\rho_1^3} y - \frac{m}{\rho_2^3} y$$

$$\frac{\partial \cup}{\partial z} = -\frac{1-m}{\rho_1^3} z - \frac{m}{\rho_2^3} z$$

Equations (4.1), (4.2) and (4.3) reduces to

$$\ddot{x} - 2\dot{y} = \frac{\partial \cup}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial \cup}{\partial y}$$

$$\ddot{z} = \frac{\partial \cup}{\partial z}$$

$$\therefore (5.1) \times 2\dot{x} + (5.2) \times 2\dot{y} + (5.3) \times 2\dot{z}$$

$$\Rightarrow 2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z} = 2\dot{x}\frac{\partial U}{\partial x} + 2\dot{y}\frac{\partial U}{\partial y} + 2\dot{z}\frac{\partial U}{\partial z}$$

$$\Rightarrow \frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = 2\frac{dU}{dt}$$

$$\Rightarrow \frac{d}{dt}\upsilon^2 = 2\frac{dU}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\upsilon^2 - 2U\right) = 0$$

$$\Rightarrow \upsilon^2 - 2U = -c^2$$

$$\Rightarrow \upsilon^2 - 2U = -c^2 \quad \text{------}(6)$$

Where υ is the speed of the infinitesional motion.

The equation (6) involves one arbitrary constant. THerefore (6) is an integral of the equations of motion. Hence to know the complete solution of the equations of motion, another five integrals are to be obtained. By further restricting the motion of the infinitesional mass to the xy-plane it is possible to reduce the number of constants required to three. Jacobi has shown that two of these are related to the third.

Ultimately, therefore for a complete solution there remaining to be found one new integral. Bruns has demonstrate that no new algebric integrals in rectangular co-ordinate exist. The equation (6) is very useful in discussing the behaviour of the infinitesional particle.

For simplification of discussion, let the infinitesional mass move in the xy-plane. The equation (6) shows that υ is a function of (x, y). The constant C depends upon the initial position and velocity of the particle.

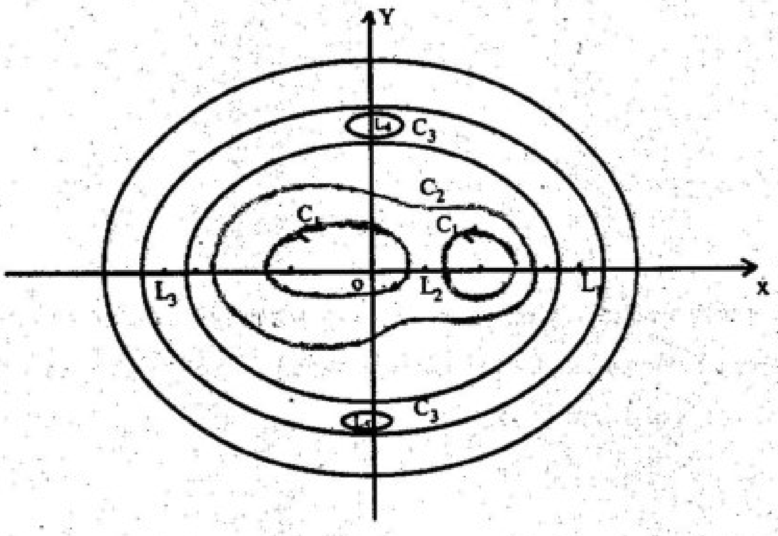
The curves of zero speed are given by $\upsilon = 0$

$$\Rightarrow 2\mu - C = 0$$

$$\Rightarrow x^2 + y^2 + \frac{2(1-m)}{\sqrt{(x - x_1)^2 + y^2}} + \frac{2m}{\sqrt{(x - x_1)^2 + y^2}} = C \quad \text{--------}(7)$$

Motion of the particle occur only in those regions of the xy-plane for which $\upsilon^2 > 0$ $\Rightarrow 2\mu - C = 0$. The curve represented by the eq_n (7) mark the boundaries of the regions within which motion can take place.

A few of contour curves are sketched in figure below.



We have taken $C_1 > C_2 > C_3$.

Case I : When C is very large $2U - C = v^2$ will be positive if x, y are very large or

$\rho_1 = \sqrt{(x - x_1)^2 + y^2}$ or $\rho_2 = \sqrt{(x - x_2)^2 + y^2}$ are very small. When x, y are very large, the curve c_1 asymotically appraches the boundary circles. For small value of P_1, P_2 the ovals C_1,C_2 sourrounding the masses 1-m and m become small in size. Motion of the particle can occur if it lies within this contour or outside the nearly. circular contour C_1

Case II : If we allow C to decrease, the ovals around (1–m) and m enpard and the outer contor moves towards the centre of the figure. In this case, the oval contours merged into a single closed contour around the two masses. The motion of the particle can occur if it lies within this contour marked C_3

Case III : If C is decreased further, the regions of stability that is, the areas of the plane in which motion can occur, become larger. The enlarged oval pattern around the finite masses marges into that outside the enterior oval leaving only a small region enclosed by C_3 where the motion is impossible.

Problem : Let $x = \gamma \cos\theta$, $y = \gamma \sin\theta$ denote polar co-ordination in the plane. Show that the equations for purturbative motion in two diminsions become

$$\ddot{r} - r\dot{\theta}^2 + \frac{k^2 m}{r^2} = \frac{\partial R}{\partial r}$$

$$\frac{d}{dt}\left(r^2\theta\right) = \frac{\partial R}{\partial \theta}$$

where $R = k^2 m' \left\{\frac{1}{\rho} - \frac{r}{r'^2}\cos(\theta - \theta')\right\}$

Solution : Let the dominat mass m_1 of the group of three masses be placed at the origin of a cortession co-ordinate system. Let m denote the body whose motiion is to be studied and m' be a mass disturbing the motion of around m_1. Let (x, y, z) and (x', y', z') be respectively the position of m & m' at time t. The equation of motion of m are

$$\ddot{x} = -\frac{k^2 m x}{r^3} + \frac{\partial R}{\partial x}$$

$$\ddot{y} = -\frac{k^2 m y}{r^3} + \frac{\partial R}{\partial y}$$

$$\ddot{z} = -\frac{k^2 m z}{r^3} + \frac{\partial R}{\partial z}$$

where $R = k^2 m' \left\{\frac{1}{\rho} - \frac{xx' + yy' + zz'}{r'^3}\right\}$

$$\rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$$

$$r^2 = x'^2 + y'^2 + z'^2$$

and $M = m + m_1$

We have $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \ \ddot{\bar{r}} = \hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}$$

$$= -\frac{k^2 m}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) + \hat{i}\frac{\partial R}{\partial x} + \hat{j}\frac{\partial R}{\partial y} + \hat{k}\frac{\partial R}{\partial z}$$

$$\ddot{\bar{r}} = -\frac{k^2 m}{r^3}\bar{r} + \bar{\nabla} R \quad \text{--------- (A)}$$

(A) is the vector eqn of motion of the mass m around m_1 under the distarbance of m'. In the present case, the motion is two dimension with the z plane as the plane of the motion. Let (r, θ) be the polar co-ordinate of (x, y) relative to O pole and OX as the initialline.

we have

$$\bar{r} = r\hat{r}$$

$$\ddot{\bar{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\theta} \quad \text{--------} (2.2)$$

$$\bar{\nabla}R = \hat{r}\frac{\partial R}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial R}{dt} \quad \text{--------} (2.3)$$

$$\left.\begin{array}{l} x = r\cos\theta,\ y = r\sin\theta,\ z = 0 \\ x = r'\cos\theta',\ y' = r\sin\theta',\ z' = 0 \end{array}\right\} \quad \text{-------} (2.4)$$

On substitution of (2.1), (2.2), (2.3) into the eq^n (A) leads to the following equations

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k^2M}{r^3} + \frac{\partial R}{\partial r}$$

$$= -\frac{k^2M}{r^2} + \frac{\partial R}{\partial r} \quad \text{-----} (3.1)$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{1}{r} + \frac{\partial R}{\partial \theta}$$

$$\Rightarrow \frac{d}{dt}\left(r^2\dot{\theta}\right) = \frac{\partial R}{\partial \theta} \quad \text{-------} (3.2)$$

where $R = k^2m'\left\{\frac{1}{P} - \frac{xx' + yy'}{r'^3}\right\}$

$$= k^2m'\left\{\frac{1}{P} - \frac{r\cos\theta\, r'\cos\theta + r\sin\theta\, r'\sin\theta'}{r'^3}\right\}$$

$$= k^2m'\left\{\frac{1}{P} - \frac{r\cos\theta\, r'\cos\theta + r\sin\theta\, r'\sin\theta'}{r'^3}\right\}$$

$$= k^2m'\left\{\frac{1}{P} - \frac{r\cos\theta\,(c\theta - \theta')}{r'^3}\right\}$$

Exercise

1. Discuss the equilateral triangle solution of the three body problem.
2. Discuss the Path for the masses corresponding to one equilateral triangle solution.

●●●

Unit-3

Perturbation

The deviation from the motion of two-body problem due the prence of other bodies or atmospheric friction (drag) or oblateness of the earth (for motion of satellite) is called the perturbation.

A modifacation in the mathematical structure of a problem changing the problem from one that can be should exactly, the unperturbed problem, to one, the pertubed problem for which it is usually possible to obtain only an approximate solution. The methods employed for this purpose from perturbation theory. These methods attempt to express the solution of the perturbed problem in the terms of the properties of the solution of the unperturbed problem.

Example of perturbation problems can be found in nearly every branch of mathematics and physics, and astronomy. The simplest case occurs in ordinary algebra. Suppose that the roots of the equation $f(x) = 0$ are known (the unperturbed problem), and that the roots of the equation $f(x) + \in g(x) = 0$ are to be found (the perturbed problem). The parameter $\in$ is here measures the size of the perturbation.

Examples of perturbations :

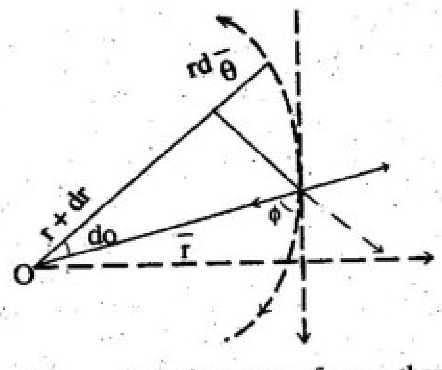
Some of variations in the orbital parameters caused by perturbations can be understood in simple terms. The lunar orbit is inclined to the ecliptic plane by about 5^0, and the longitude of its ascending node on the ecliptic plane is observed to regress a complete revolution in 1861 years.

3.2 Equation of the Orbit :

For the motion of the earth's satellite the atmospheric arising out of scalar winds and obletness of the earth's topography may be taken into account depending upon some physical situations. Futher, the perturbation on the motion of a satellite can be taken in tangential direction for atmospheric effect (or drag)

Let us consider the motion of a mass 'm' in the Newtonian field of gravitation exerted due to the mass 'M'.

The tangential force $R = \frac{cV}{r^2}$



Where V is the orbital speed and r is the distance from the centre of mass, then the components of acclerations are $-R\cos\phi$ and $-R\sin\phi$

$$\text{or} - \frac{cv}{r^2}\frac{d\gamma}{ds} \text{ and } - \frac{cv}{r^2}\frac{d\theta}{ds}$$

$$\text{or} - \frac{c}{r^2}\frac{ds}{dt}.\frac{d\gamma}{ds} \text{ and } - \frac{c}{r}\frac{ds}{dt}.\frac{d\theta}{ds}$$

$$\text{or} - \frac{c}{r^2}\frac{d\gamma}{dt} \text{ and } - \frac{c}{r}\frac{d\theta}{dt}$$

$\therefore$ The eq^n of motion in a plane is given by

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} - \frac{c}{r^2}\dot{r} \text{ ------ (i)}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = -\frac{c}{r}\dot{\theta} \text{ ------ (ii)}$$

Integrating equation (ii) we get

$$\left(r^2\dot{\theta}\right) = H = h - r\theta \text{ ------ (iii)}$$

where h is the integration constant but $H\left(= r^2\dot{\theta}\right)$ is not constant.

Let us put $u = \frac{1}{r}$ so that $\left(r = \frac{1}{u}\right)$

$$\dot{r} = \frac{d\gamma}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}.\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}.\frac{H}{r^2}$$

$$= -H\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d}{d\theta} = \left(-H\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -H\left(\frac{d^2u}{d\theta^2} + \frac{dH}{d\theta}.\frac{du}{d\theta}\right)\frac{H}{r^2} \qquad \because \frac{dH}{d\theta} = -C$$

$$= -H^2u^2\frac{d^2u}{d\theta^2} + CHu^2\frac{du}{d\theta}$$

,(using (iii))

(i) reduces to

$$\Rightarrow -H^2u^2\frac{d^2u}{d\theta^2} + HCu^2\frac{du}{d\theta} = -\mu u^2 + Cu^2H\frac{d\mu}{d\theta} - \frac{1}{u}H^2u^4$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu}{H^2}$$

Which is the differential eqn of the orbit with variable $H = h - c\theta$.

Now

$$\frac{\mu}{H^2} = \frac{\mu}{(h - c\theta)^2}$$

$$= \mu (h - c\theta)^{-2}$$

$$= \mu h^{-2} \left(1 - \frac{c\theta}{h}\right)^{-2}$$

$$= \frac{\mu}{h^2}\left(1 + 2\frac{c\theta}{h}\right) \quad \Big| \quad \because \left|\frac{c\theta}{h}\right| < 1$$

Since for small c, the higher order terms are neglected.

$\therefore$ The differential eq^n of the orbit takes the from

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\left(1 + \frac{2c\theta}{h}\right)$$

$\therefore$ The solution of the equation for the orbit is given by

$$u = A\cos(\theta - \omega) + \frac{\mu}{h^2}\left(1 + \frac{2c\theta}{h}\right)$$

$$\Rightarrow u = \frac{\mu}{h^2}\left[\frac{Ah^2}{\mu}\cos(\theta - \omega) + 1 + \frac{2c\theta}{h}\right]$$

$$\Rightarrow \frac{h^2}{\mu} u = 1 + e\cos(\theta - \omega) + 1 + \frac{2c\theta}{h} \quad \text{----- (iv)}$$

where $e = \dfrac{Ah^2}{\mu}$

Which is the equation of the orbit but it is not an ellipse due to the presence of the term $\left(\dfrac{2c\theta}{h}\right)$

Letus assume that at certain position at point $P_1(\mu_1, \theta_1)$ having velocity $\left[\left(\dfrac{d\mu}{dt}\right)P_1' \left(\dfrac{d\theta}{dt}\right)P_1\right]$ the perturbation force auses to act.

$$\therefore u = \frac{1}{\ell_1}\left[1 + e_1\cos(\theta_1 - \omega_1)\right] \quad \text{----- (v)}$$

Where suffix '1' is used to indentify the elliptic orbit when the perturbative effect is absent.

In this case $\ell_1 = \dfrac{h_1^2}{\mu}$ with $h_1 = h - c\theta$,

$$= \frac{1}{\mu}(h - c\theta_1)^2$$

$$= \frac{h^2}{\mu}\left(1 - \frac{c\theta}{h}\right)^2$$

$$= \frac{h^2}{\mu}\left(1 - \frac{2c\theta_1}{h}\right)$$

Since for C, the higher order term are neglected.

$$\because \ell_1 = \ell\left(1 - \frac{2c\theta_1}{h}\right)$$

Now at the point $P_1(\mu_1, \theta_1)$ both μ and $\left(\frac{d\mu}{d\theta}\right)$ must be same.

$$(u_1 =) = \frac{1}{\ell_1}[1 + e_1\cos(\theta_1 - \omega_1)] = \frac{1}{\ell}\left[1 + e\cos(\theta_1 - \omega_1) + \frac{2c\theta_1}{h}\right]$$

$\left[\because (u_1, \theta_1) \text{ is same for both the orbit}\right]$

$$\text{or } \frac{1}{e\left(1 - \frac{2c\theta_1}{h}\right)}[1 + e_1\cos(\theta_1 - \omega_1)] = \frac{1}{e}\left[1 + e\cos(\theta_1 - \omega) + \frac{2c\theta_1}{h}\right]$$

$$\text{or } [1 + e_1\cos(\theta_1 - \omega_1)] = \left(1 - \frac{2c\theta_1}{h}\right)\left[1 + e\cos(\theta_1 - \omega) + \frac{2c\theta_1}{h}\right]$$

$$\text{or } 1 + e_1\cos(\theta_1 - \omega_1) = 1 + e\cos(\theta_1 - \omega_1)\ \frac{2c\theta_1}{h} - \frac{2c\theta_1}{h} - \frac{2c\theta_1}{h}\ e\cos(\theta_1 - \omega) - \frac{4c^2\theta_1^2}{h^2}$$

$$\therefore e_1\cos(\theta_1 - \omega_1) = e\cos(\theta_1 - \omega_1) - \frac{2ce\theta_1}{h}\cos(\theta_1 - \omega) \text{ ----- (vi)}$$

Also

$$\left[\left(\frac{du}{d\theta}\right)_{P_1} = \right] \frac{1}{\ell_1}[-e_1\sin(\theta_1 - \omega_1)]$$

$$= \frac{1}{e}\left[-e\sin(\theta_1 - \omega) + \frac{2c}{h}\right]$$

Differentation (iv) and (v) w.r.t. θ then putting (μ_1, θ_1)

$$\Rightarrow -e_1\sin(\theta_1 - \omega) = \left(1 - \frac{2c\theta_1}{h}\right)\left[-e\sin(\theta_1 - \omega) + \frac{2c}{h}\right]$$

$$= -e\sin(\theta_1 - \omega) + \frac{2c}{h} + \frac{2c\theta_1}{h}e\sin(\theta_1 - \omega) + 0$$

$$\Rightarrow e_1 \sin(\theta_1 - \omega) = e \sin(\theta_1 - \omega) - \frac{2c}{h} - \frac{2c\theta_1}{h} e \sin(\theta_1 - \omega) \quad \text{----- (vii)}$$

(vi) $\times \cos(\theta_1 - \omega)$ + (vii) $\times \sin(\theta_1 - \omega)$ gives

$$e_1 \left[\cos(\theta_1 - \omega) \cos(\theta_1 - \omega_1) + \sin(\theta_1 - \omega) \sin(\theta_1 - \omega_1)\right]$$

$$= e \cos^2(\theta_1 - \omega) - \frac{2c\theta_1}{h} e \cos^2(\theta_1 - \omega) + e \sin^2(\theta_1 - \omega)$$

$$- \frac{2c}{h} \sin(\theta_1 - \omega) - \frac{2c\, e\, \theta}{h} \sin^2(\theta_1 - \omega)$$

$$\Rightarrow e_1 \left[\sin(\theta_1 - \omega_1 - \theta_1 + \omega)\right] = e - \frac{2c\theta_1 e}{h} - \frac{2c}{h} \sin(\theta_1 - \omega) \quad \text{----- (viii)}$$

$$\Rightarrow e_1 .1 = e - \frac{2c\theta_1 e}{h} - \frac{2c}{h} \sin(\theta_1 - \omega)^{xy}$$

$$\Rightarrow e_1 - e = - \frac{2c\theta_1 e}{h} - \frac{2c}{h} \sin(\theta_1 - \omega)$$

$$\Rightarrow \partial e_1 = - \left[\frac{2c\theta_1 e}{h} + \frac{2c}{h} \sin(\theta_1 - \omega) \right] \quad \text{----- (ix)}$$

Again (vi) $\times \sin(\theta_1 - \omega)$ - (vii) $\times \cos(\theta_1 - \omega)$

$$\Rightarrow e_1 \partial \omega_1 = \frac{2c}{h} \cos(\theta_1 - \omega) \quad \text{----- (x)}$$

From (ix), it is sun that there two parturbative effect, one is purly linear called secular perturbation and other is called the periodic perbutation.

The effect of periodic perturbation is negligible in the long run but the effect of secular perturbation through small can effect the system considerably.

Also from (x) the parturbation of langitude (ω) is fully periodic, so it can be ignored.

Let $\Delta \ell_1$, $\Delta \eta_1$ and Δa_1 be the parturbative effect in ℓ, η and a respectively.

Now $\ell_1 = \ell \left(1 - \frac{2c\theta_1}{h}\right)$

$$\log \ell_1 = \log \ell + \log \left(1 - \frac{2c\theta_1}{h}\right)$$

$$= \log \ell + \left(-\frac{2c\theta_1}{h}\right) + 0.\left(c^2\right)$$

[order of C^2 has been neglected]

Taking differential

$$\frac{\Delta \ell_1}{\ell_1} = -\frac{2C}{h}\Delta\theta_1$$

$$\text{or } \frac{\Delta \ell_1}{\ell_1} = -\frac{2C}{h}2\pi \qquad \text{for one revolution } \theta_1 = 2\pi$$

$$= -\frac{4C\pi}{h}$$

Again

$$\ell_1 = a_1\left(1-e_1^2\right) \qquad \because \ell = \frac{b^2}{a}$$

$$\log \ell_1 = \log a_1 + \log\left(1-e^2\right) \qquad = \frac{a^2\left(1-e^2\right)}{a} = a\left(1-e^2\right)$$

$$\frac{\Delta \ell_1}{\ell_1} = \frac{\Delta a_1}{a} + \frac{-2e_1^2 \Delta e_1}{\left(1-e_1^2\right)e_1} \quad \text{----- (xi)}$$

From (viii) $e_1 = e - \frac{2ce\theta_1}{h} - \frac{2c}{h}\sin(\theta_1 - \omega)$

$$= e\left(1 - \frac{2c}{h}\theta_1\right)$$

[$\because$ Periodic perturbation can be neglected]

$$\log e_1 = \log e + \log\left(1 - \frac{2c\,\theta_1}{h}\right)$$

$$= \log e + \log\left(-\frac{2c\,\theta_1}{h}\right) + 0\left(c^2\right)$$

$$= \log e - \frac{2c\,\theta_1}{h}$$

$$\therefore \frac{\Delta e_1}{e_1} = -\frac{2C}{h}2\pi \qquad [\because \text{for one revolution } \theta_1 = 2\pi]$$

$$\therefore \frac{\Delta e_1}{e_1} = -\frac{4C\pi}{h} = \frac{\Delta e_1}{e_1}$$

From (xi), we get

$$\frac{\Delta \ell_1}{\ell_1} = \frac{\Delta a_1}{a_1} + \frac{-2e_1^2}{\left(1-e_1^2\right)}\left(-\frac{4c\pi}{h}\right)$$

$$\therefore \frac{\Delta a_1}{a_1} = -\frac{4c\pi}{h} - \frac{-2e_1^2 \times \frac{4c\pi}{h}}{1-e_1^2}$$

$$= -\frac{4c\pi}{h}\left[1 + \frac{-2e_1^2}{1-e_1^2}\right]$$

$$= -\frac{4c\pi}{h}\frac{1 e_1^2}{1-e_1^2} \quad \text{-----} \ (xii)$$

Also from

$$\eta^2 a^3 = \mu$$

we have $\eta_1 = \sqrt{\mu}\, a_1^{-3/2}$

$$\therefore \log \eta_1 = \log\sqrt{\mu} + \left(-\frac{3}{2}\right)\log a_1$$

$$\therefore \frac{\Delta \eta_1}{\eta_1} = -\frac{3}{2} \times \frac{\Delta a_1}{a_1}$$

$$= -\frac{3}{2} \times \left(-\frac{4\pi c}{h}\frac{1+e_1^2}{1-e_1^2}\right)$$

$$= \frac{4\pi c}{h}\frac{1+e_1^2}{1-e_1^2} \quad \text{-----} \ (xiii)$$

From (xii) and (xii) it is sun that perturbation in a decreases and that in 'n' increases even for small values of e_1

This these changes are called the "variation" of the elements of the turn orbit e and ω.

Which are integrational constant.

Problem : Determine the expression of ecentric anomaly in terms of e (eccentricity) and m (mean anomaly) in series from as

$$\phi = m + \left(e - \frac{e^3}{8}\right)\sin m + \frac{1}{2}e^2 \sin 2m + \frac{3}{8}e^3 \sin 3m$$

upto 3rd degree of e.

Solution : From Keplir's equation, we have

$$m = \phi - e\sin\phi$$

$$\text{or } \phi = m + e\sin\phi$$

Some $e < 1$ and $\sin\phi < 1$ therefore the first approximation ϕ_1 can be considered as $\phi_1 = m$

If ϕ_2 is 2nd order approximation of ϕ, then

$$\phi_2 = m + e \sin \phi_1$$

$$= m + \ell \sin m$$

Again ϕ_3 is third order approximation, then

$$\phi_3 = m + e \sin \phi_2$$

$$= m + e \sin (m + e \sin m)$$

$$= m + e \{\sin \cos e (\sin m) + \cos m \sin (e \sin m)\}$$

$$\therefore \phi_3 = m + (e \sin m . 1 + e \cos m. e\sin m)$$

$$\because \sin\theta = \theta \text{ and } \cos\theta = 1$$

as θ is small.

$$= m + e \sin m + \frac{1}{2} e^2 \sin 2 m$$

Again considering the next approximation we can write

$$\phi_4 = m + e \sin \phi_3$$

$$= m + e \sin \left(m + e \sin m + \frac{1}{2} e^2 \sin 2 m\right)$$

$$= m + e \sin m \cos\left(e \sin m + \frac{1}{2} e^2 \sin 2 m\right) + e \cos m \sin\left(e \sin m + \frac{1}{2} e^2 \sin 2 m\right)$$

$$\phi_4 = m + e \sin m \left\{1 - \frac{1}{2} e^2 \sin^2 m\right\} + e \cos m \times \left\{e \sin m + \frac{1}{2} e^2 \sin 2 m\right\}$$

$$= m + e \sin m - \frac{1}{2} e^3 \sin^3 m + \frac{1}{2} e^2 \sin 2 m + \frac{1}{2} e^3 \sin 2 m \cos m$$

$$= m + e \sin m - \frac{1}{8} e^3 (3\sin m - \sin 3m) + \frac{1}{2} e^2 \sin 2 m + \frac{1}{4} e^3 (\sin 3 m + \sin m)$$

$$\phi = m + \left(e - \frac{e^3}{8}\right) \sin m + \frac{1}{2} e^2 \sin 2 m + \frac{3}{8} e^3 \sin 3 m.$$

Exercise

1. Deduce the equation of orbit as

$$\frac{\ell}{r} 1 + e \cos (\theta - \omega) + \frac{2 c \theta}{h}$$

under tangential resistance

$$R = \frac{CV}{r^2}$$

with usual meanings of the symbols. What are secular and periodic perturbations?

2. Show that the components F_r, F_b, F_n of the perbutating force $\bar{F}$ on the motion of moon relative to the earth due to the sun are

$$F_r = \frac{1}{2} r N^2 \left[1 + 3\cos 2(u-U)\right]$$

$$F_b = -\frac{3}{2} r N^2 \sin 2(u-U)$$

$$F_n = -3 r N^2 \sin i \sin U \cos (u-U)$$

where $N^2 = \frac{GM}{R^3}$

3. What do you mean by osculating orbit of perturbative motion? Determine in this case, the magnitude of perturbation.

4. From the orbit

$$\frac{\ell}{r} 1 + e\cos(\theta - \omega) + \frac{2c\theta}{h}$$

under small tangential perturbation, deduce the expressions

$$\partial e_1 = -\frac{2c}{h} + \left[e\,\theta_1 + \sin(\theta_1 - \omega)\right]$$

and

$$e_1\,\partial\omega_1 = -\frac{2c}{h}\cos(\theta_1 - \omega)$$

to give secular and periodic perturbations the symbols have their usual meaning.

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UNIT- 4

Flight Machanics

4.1. Defination of Rocket :

A variable mass vihicle where a part of its mass is lost by high speed moving particles due to fuel burning causing upward thrust is called a Rocket.

A roket is used to launch a vihicle or a satellite in to its desired orbit in space for various purposes. Depending upon the hight of projection it is necessary to take the help of multi-stage rocket. Actually, the satellite or space is to be projected to certain hight to undertake some orbit based on the projection. If a satellite is to be placed in the gravitional region of the earth, then it must be imparted minimum initial velocity

$V = 7.6$ km / sec.

4.2. Equation of motion variable mass :

A variable mass of a system may continously increase due to coalesce or decrease due to ejection of particle usually for burning fuels. For space vehicles due to ejection of matter the mass decreases.

But due to coalesce a rain drop may acquire additional masses from dust particles.

Let m be the mass of the body moving with velocity V at time t. Let after small time interval ∂t it encounter with small mass '∂m' moving with velocity u which coalesce with m.

Let $(v + \partial v)$ be the resultant velocity of the combined mass at time $(t + \partial t)$. But at the time of coalesce, there may appear interacting forces T (say) to act with the external force F arising out of gravity and and air resistance (or drag).

$\therefore$ T = change of lenear momentum of the small mass ∂m

$$= \lim_{\partial t \to 0} \frac{\partial m (v + \partial v) - \partial m\, u}{\partial t}$$

$$= \lim_{\partial t \to 0} (V - u) \frac{\partial m}{\partial t} \qquad \because \partial m\, \partial v << 1 \text{ as } \partial t \to 0$$

$$= (V - u) \lim_{\partial t \to 0} \frac{\partial m}{\partial t}$$

$$= (V - u) \frac{dm}{dt}$$

Again the change in linear momentum of the mass is given by

$$= \lim_{\partial t \to o} \frac{m(v + \partial v) - mv}{\partial t}$$

$$= \lim_{\partial t \to o} m \frac{\partial v}{\partial t}$$

$$= m \cdot \frac{dm}{dt}$$

∴ The equation of variable mass is given

$$m \frac{dv}{dt} = F - T$$

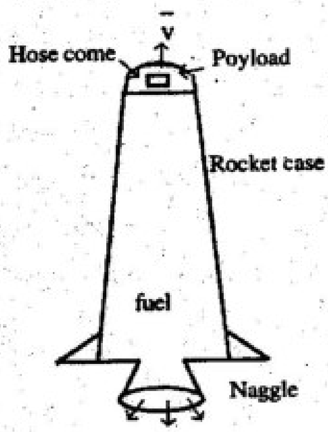
$$\Rightarrow F - T = m \frac{dv}{dt}$$

$$\Rightarrow F = m \frac{dv}{dt} + T$$

$$= m \frac{dv}{dt} + (V - u) \frac{dm}{dt}$$

$$= \frac{dv}{dt}(m v) - u \frac{dm}{dt}$$

4.3. Performance of single stage rocket and equation for the satillite in vacuum (without gravity)



Let us consider a first stage rocket consisting of a Nose cone at the top which has housed the payload (or satellite) and a nozzle at the bottom to release exhaust particles for upward thrust. An amount of fuel is inserted in the metalic case of the satelite besides the pay-load which is prottected from atmospheric effect by the nose cone.

If $\bar{v}$ is the velocity of the rocket and it is the velocity of the burning fuel, then the eqn motion mass is

$$\bar{F} = m \frac{dv}{dt} + (\bar{v} - \bar{u}) \frac{dm}{dt}$$

If ϑ_e is the relative velocity of the exhaust particles coming out of the rocket then

$$V - \mu = \vartheta_e$$

$$\therefore F = \frac{dv}{dt} + \vartheta_e \frac{dm}{dt}$$

If the first consideration we ignore the external force (or gravity) so that

$$m \frac{dv}{dt} = -\vartheta_e \frac{dm}{dt} \text{ ----- (1)}$$

Let M be the total (composite) mass of the rocket so that

$$M = M_p + M_f + M_s$$

where M_p is the mass of the payload (or satellite), $M_f(o)$ is the mass of the initial amount of fluid and M_s is the mass of the matelic stracture.

Let us take

$$M_f(o) + M_s = M_o \text{ and}$$

$$M_f(o) = \in M_o \text{ so that } M_s = (1 - \in) M_o$$

Where is called the structural factor.

The amount of fuel to be include depends on the value of $\in$, $0 < \in < 1$ and the mass of the structure.

∴ The equation o. ..-'otion takes the form

$$M \frac{dv}{dt} = \vartheta_e \frac{dM}{dt} \text{ ----- (2);}$$

with M given by equation (1)

Let us consider that the fuel burn at the constant rate K (say)

If $M_f(t)$ is the mass of fuel at any time then

$$\frac{d}{dt} = [M_f(t)] = -K$$

$$\therefore \int_{t=o}^{t} d[M_f(t)] = -K\int_{o}^{t} dt$$

$$\Rightarrow M_f(t) - M_f(o) = -Kt$$

$$\Rightarrow M_f(t) = -Kt + M_f(o)$$

$$\Rightarrow M_f(t) = -Kt + \in M_o \text{ ----- (3)}$$

Let us assume that the amount of fuel completely burns at $t = t_b$

From (3)

$$M_f(t_b) = o = M_f(o) - Kt_b$$

$$\Rightarrow \in M_o = Kt_b \text{ ------ (4)}$$

Hence

$$M = M_P + M_f(t) + (1-\in) M_o$$

$$= M_P + (M_f(o) - kt) + (1-\in) M_o$$

$$= M_P + \in M_o - kt + (1-\in) M_o$$

$$= M_P + M_o - kt$$

$\therefore$ The eq^n of motion $(M_P + M_o - kt)\dfrac{d\bar{v}}{dt}\ \vartheta_e k$

$$\left(\because d\frac{M_f}{dt} = \frac{dM}{dt} = -k\right)$$

$$\text{or } d\bar{v} = \frac{\vartheta_e k}{M_P + M_o - kt}\,dt$$

$$\int_0^{t_b} dv = \int_0^{t_b} \frac{\vartheta_e k}{M_P + M_o - kt}\,dt$$

$$\Rightarrow V(t_b) - V(o) = -\vartheta_e\left[\log(M_P + M_o - kt)\right]_o^{t_b}$$

$$= -\vartheta_e\left[\log(M_P + M_o - kt_b) - \log(M_P + M_o)\right]$$

$$= -\vartheta_e \log\left(\frac{M_P + M_o - kt_b}{M_P + M_o}\right)$$

$$= -\vartheta_e \log\left[1 - \frac{\in M_o}{M_P + M_o}\right] \quad \because \in M_o = kt_b$$

$$= -\vartheta_e \log\left[1 - \frac{\in}{1 + M_P/M_o}\right]$$

Which gives change (increment) on velocity to the payload launched by first stage rocket.

It is sun that the performance of the first stage rocket depends on

(i) The exhaust velocity ϑ_e

(ii) The structural factor $(1 - \in)$ due to $\in$

(iii) The mass ratio $\frac{M_P}{M_o}$, a constant.

4.4. When the external porce is taken on the gravity in the performance of the first stage rocket.

If one gravity is considered as the external force then the equation of motion, takes the form

$$(M_P + M_o - kt)\frac{d\bar{v}}{dt} = -\vartheta_e k (M_P + M_o - kt) - g$$

$$\text{or } \frac{d\bar{v}}{dt} = \frac{\bar{\vartheta}_e k}{M_P + M_o - kt} - g$$

$$\text{or } \int_0^{t_b} d\bar{v} = -\bar{\vartheta}_e \int_0^{t_b} \frac{k\,dt}{M_P + M_o - kt} - \int_0^{t_b} g\,dt$$

Where t_b is the time of complete fuel burn

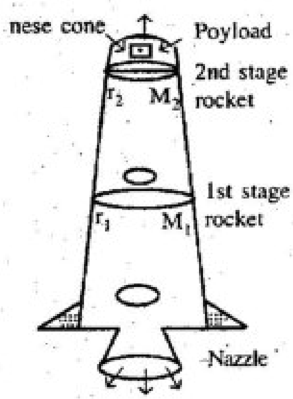
$$\text{or } \bar{v}(t_b) - \bar{v}(o) = -\vartheta_e \left[\log (M_P + M_o - kt)\right]_0^{t_b} - g^{t_b}$$

$$\text{or } \Delta\bar{v} = -\vartheta_e \left[\log \frac{M_P + M_o - k}{M_P + M_o}\right] - g^{t_b}$$

$$\text{or } \Delta\bar{v} = -\vartheta_e \left[\log\left(1 - \frac{\in M_o}{M_P + M_o}\right)\right] - g\frac{\in M_o}{K}$$

Which is the incrimental change in velocity of the rocket due to the expense of puel burning.

4.5. Performance of the 2nd stage rocket :



A simple model of two stage rocket consis of two matelic cylinderical structures with the pay-load of the nose cone attached to the 2nd chamber and the nozzle to release exhaust particles with high speed at the bottom.

Let M_1 and M_2 be the masses of the 1st and 2nd chamber respectively. Also let t_1 & t_2 be the respective times of fuel burning in the moter equipped with fuels.

We know the velocity after the 1st stage rockets is

$$\left(\Delta \bar{v} =\right) \bar{v}_1 = -\vartheta_e \log\left[1 - \frac{\in M_o}{M_P + M_o}\right] \quad \text{----- (1)}$$

Where ϑ_e is the exhaust speed. It needs to report that after complete burning of the fuel of the 1st stage the structure is detached and the ignition of the fuel of 2nd stage starts functioning. For convenionce, let us assume the exhaust speed for the 2nd stage rocket be also ϑ_e.

Clearly after the ditachment of the first stage. The 2nd stage rocket moves with initial velocity V_1 given by (1)

$$\int_0^{t_2} d\bar{v} = -\bar{\vartheta}_e \int_0^{t_2} \frac{kdt}{M_P + M_2 - kt}$$

$$\text{or } \bar{v}(t_2) - \bar{v}(o) = -\bar{\vartheta}_e \left[\log\left(M_P + M_2 - kt\right)\right]_o^{t_2}$$

$$\text{or } \bar{v}(t_2) = -\vartheta_e \log\left[\frac{M_P + M_2 - kt}{M_P + M_2}\right] + \bar{v}_1$$

$\therefore$ The initial velocity $\bar{v}(o) = \bar{v}_1$

$$\text{or } \bar{v}(t) = -\vartheta_e \log\left[1 - \frac{kt}{M_P + M_2}\right] + v_1$$

$$= -\vartheta_e \log\left[1 - \frac{\in M_2}{M_P + M_2}\right] - \vartheta_e \log\left[1 - \frac{\in M_1}{M_P + M_1 + M_2}\right]$$

Which gives the performence of 2nd stage rocket producing there by ligher imparted velocity.

4.6. Problem :

Example 1.

A rocket of total mass μ containing *a* proportion $(0 < \in < 1)$ as fuel. If the exhuast speed is ϑ_e, *a* constant and the mass of the pay-load is ignored prove that the final velocity of the satellite is independent of the rate of fuel burn

$$\left[(M_P + M_o - kt)\frac{dv}{dt} = \vartheta_e k - (M_P + M_o - kt) g\right]$$

Solution : Since the total mass of the system is M, therefore the equation of motion (in absence of gravity) is given by

$$(M_P + M_o - kt)\frac{dv}{dt} = \vartheta_e k$$

or $\frac{dv}{dt} = \frac{\vartheta_e k}{(M_o - kt)}$

or $\int_o^{t_b} dv = \int_o^{t_b} \frac{\vartheta_e k}{(M_o - kt)} dt$

Where t_b is the time of complete burning of the fuel.

$$\therefore [V]_o^{t_b} = - \vartheta_e [\log (M_o - kt)]_o^{t_b}$$

$$\Rightarrow \Delta v = - \vartheta_e \left[\log \left\{ \frac{M_o - kt_b}{M_o} \right\} \right] \quad \text{------ (1)}$$

But we have $\frac{d}{dt} = (M_f) = - k$

$$\Rightarrow \int_o^{t_b} \frac{d}{dt} (M_f) = - \int_o^{t_b} k\, d\, t$$

$$\Rightarrow M_f (t_b) - M_f (o) = - kt_b$$

$$\Rightarrow M_f (o) = kt_b$$

Now

$$M_f (o) + M_s = M_o$$

$$\therefore k\, t_b + M_s = M_o$$

$$\Rightarrow M_o - k\, t_b = M_s$$

$$\therefore (1) \Rightarrow (V)_{t_b} = - \vartheta_f \log \left(\frac{M_s}{M} \right)$$

Here $M_o = M$

Which is the independent of fuel.

Example 2. : If M_1 and M_2 be the masses of a double stage rocket, so that $M_1 = M_2 = 50$ P, Mp being the mass of the payload $\in = 0.8$ and the specity impulse Isp = 300 sec. then show that the rocket is not capable of putting the satellite into earth bound orbit.

Solution : The specific impulse

$\Rightarrow 300$ = Trust per unit weight fuel burning per sec.

$$\Rightarrow 300 = \frac{T}{kg} \qquad \because T = \left(\bar{v} - \bar{\vartheta}\right)\frac{d\mu}{dt}$$

$$\Rightarrow 300 = \frac{\vartheta_e' k}{kg} \qquad = \vartheta_e' k$$

$$\Rightarrow \vartheta_e' = 300 \times g$$

$$= 300 \times 9.8 \text{ m/sec}$$

But the final velocity after the end of the 2nd-stage rocket is given by

$$V_2 = -\vartheta_e' \log\left[1 - \frac{\in M_1}{M_P + M_1 + M_2}\right] - \vartheta_e' \log\left[1 - \frac{\in M_2}{M_P + M_2}\right]$$

$$= -300 \times 9.8 \log\left[1 - \frac{0.8 \times 50P}{M_P + 50\,M_P + 50M_P}\right]$$

$$-9.8 \times 300 \log\left[1 - \frac{0.8 \times 50M_P}{M_P + 50\,M_P}\right] \text{m/sec}$$

$$V_2 = -2940 \log\left(1 - \frac{40}{101}\right) - 2940 \log\left(1 - \frac{40}{51}\right)$$

$$= -2940 \log\left(\frac{61}{101}\right) - 2940 \log\left(\frac{11}{51}\right)$$

$$= -2940 \times (-0.21899) + 2940 \times (0.66618)$$

$$= 2940 \times 0.21899 + 2940 \times 0.66618$$

$$= 643.566 + 1958.5692$$

$$= 5.9 \text{ km/sec.}$$

This shows that r the rocket is not capable of placing the satellite in earth bound orbit.

Exercise

1. Discuss the performance of a two-stage rocket in launching a satellite in space. [2007]

2. Discuss the performance of a single-stage rocket, deducing the equation of motion of a rocket in vacuum. A rocket of total mass M contains a proportion $\in M\ (0 < \in < 1)$ as fuel. If exhaust speed of the rocket in absence of pay-load is independent of the rate at which the fuel is burnt. [2008]

3. What do you mean by a rocket? Find the motion of a rocket in vacuum and the performance of a single-stage rocket. [2007]

4. A rocket engine of mass 3×10^5 kg without payload ejects exhaust gases with velocity 3000 m/s and at a rate 10^4 kg/s. The rocket is fired vertically from the surface of the earth. Show that such a rocket is not capable of escaping from the earth's gravitational field, assuming the gravitional force acting, on the rocket during the flight constant. [2007]

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