M1204

Institute of Distance and Open Learning
Gauhati University

M.A./M.Sc. in Mathematics
Semester 2

Paper IV
Mathematical Methods

Contents:

Course Co-Ordination

Prof. Amit Choudhury — Director, GU, IDOL

Dr. Chandra Rekha Mahanta — Associate Professor, Dept. of Mathematics, Gauhati University

Contributors:

Prof. Rudra Kanta Deka — Dept. of Mathematics, Gauhati University

Editorial Team

Prof. Kuntala Patra — Dept. of Mathematics, Gauhati University

Prof. Nanda Ram Das — Dept. of Mathematics, Gauhati University

Dipankar Saikia — Editor (SLM) G.U, IDOL

Cover Page Designing:

Kaushik Sarma : Graphic Designer, CET, IITG

Re-Print Feb., 2019

Unit 1

Integral transforms :

Let K(s,t) be a known function of the two variables s and t, then the integral transform F(s) of the function f(t) is defined by an integral of the form

$$\int_a^b K(s,t)\,f(t)\,dt = F(s)$$

where K(s,t) is called the kernel of the integral transform.

The kernel and limits of integration for various integral transforms are given as follows:

	Name of the transform	*K(s,t)*	*a*	*b*
(i)	Laplace transform	e^{-st}	0	∞
(ii)	Fourier transform	e^{ist}	$-\infty$	∞
(iii)	Hankel transform	$t\,J_n(st)$	0	∞
(iv)	Mellin transform	t^{s-1}	0	∞

We shall discuss the transforms (i), (ii) and (iv) in the succeeding chapters.

Laplace Transform method :

Laplace transform is essentially a mathematical tool which can be used to solve several problems in science and engineering. This transform was first introduced by Laplace, a French mathematician, in the year 1790. To the basic question as to why one should learn Laplace transform technique when other techniques are available, the answer is very simple. Transforms are used to accomplish the solution of certain problems with less effort and in a simple routine way. To illustrate, consider the problem of finding the value of x from the equation

$$x^{1.85} = 3$$

It is an extremely tedious task to solve this problem algebraically. However, by use of Logarithms and anti-logarithms we can solve as

$$x = \ln^{-1}\left(\frac{\ln 3}{1.85}\right)$$

With the help of any ordinary calculator, we can now compute x. Following this simple example, the Laplace transform method reduces the solution of an ordinary differential equation to the solution of an algebraic equation. In fact this method has a particular advantage in finding the solution of an ordinary differential equation with appropriate initial conditions, without first finding the general solution and then using initial conditions for evaluating the arbitrary constants. Also, when the Laplace transform technique is applied to a PDE, it reduces the number of independent variables by one.

Definition :

A function f(t) is said to be a piecewise continuous (or sectionally continuous) on a finite interval $a \le t \le b$ if this interval can be divided into a finite number of subintervals such that (i) f is continuous in the interior of these subintervals and (ii) f(t) approaches finite limit as t approaches either endpoint of each of the subintervals from its interior.

Definition :

Suppose f(t) is a piecewise continuous function and if it has an additional property that there exists a real number α and a finite positive number M such that

$$\underset{t\to\infty}{Lt}|f(t)|e^{-st} \le M \quad \text{for } s > \alpha$$

and the limit does not exists when $s < \alpha$, then such a function is said to be of exponential order α, also written as

$$|f(t)| = 0\left(e^{\alpha t}\right)$$

Variables such as velocity and current are always finite; which means that f(t) is bounded. Thus for any bounded function $f(t), |f(t)|e^{-st} \to 0$ for all s>0. The order of such a function is zero.

For illustration, let us consider the following examples:

(i) $f(t) = e^{at}\sin bt$ is of exponential order with the constant α=a, because

$$e^{-\alpha t}|f(t)| = e^{-at}e^{at}|\sin bt|$$
$$= |\sin bt|, \text{ bounded } \forall t$$

(ii) $f(t) = t^n$, $n > 0$ is of exponential order, because

$$e^{-\alpha t}|f(t)| = e^{-\alpha t}t^n$$

Now for any $\alpha > 0$, $\lim_{t\to\infty} e^{-\alpha t}t^n = 0$.

Thus, $\exists M > 0,\ t_0 > 0$ such that

$$e^{-\alpha t}|f(t)| = e^{-\alpha t}t^n < M \quad \text{for } t > t_0$$

Hence t^n is of exponential order with the constant α equal to any positive integer.

(iii) $f(t) = e^{t^2}$ is not of exponential order because

$$e^{-\alpha t}|f(t)| = e^{t^2-\alpha t}$$

and this become unbounded as $t \to \infty$, no matter what is the value of α.

Definition :

Let f(t) be a continuous and single valued function of the real variable t defined for all t, $0 < t < \alpha$, and is of exponential order. Then the Laplace transform of f(t)is defined as a function F(s) denoted by the integral

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

over that range of values of s for which the integral exists. Here, s is a parameter, real or complex. Obviously L{f(t)} is a function of s. Thus

$$L\{f(t)\} = F(s)$$
$$f(t) = L^{-1}\{F(s)\}$$

where L is the operator which transforms f(t)into F(s), called Laplace transform operator, and L^{-1} is the inverse Laplace transform operator.

Now we are in a position to verify the following important result.

Theorem 1 :

If f(t) is piecewise continuous in the range $t \geq 0$ and is of exponential order α, then the Laplace transform F(s) of f(t) exists for all $s > \alpha$.

Proof : From the definition of Laplace transform

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{T} e^{-st} f(t)\,dt + \int_T^{\infty} e^{-st} f(t)\,dt$$

$$= I_1 + I_2$$

Since f(t) is piecewise continuous on any finite interval 0<t<T, I_1 exists, whereas

$$|I_2| \leq \int_T^{\infty} \left| e^{-st} f(t) \right| dt$$

But f(t) is a function of exponential order, therefore

$$|f(t)| < Me^{\alpha t} \text{ for } \alpha \text{ real}$$

Hence $\quad \left|e^{-st} f(t)\right| < Me^{-(s-\alpha)t}$

Thus

$$|I_2| = \int_T^{\infty} e^{-(s-\alpha)t} M\,dt = \frac{Me^{-(s-\alpha)T}}{s-\alpha}, \quad s > \alpha$$

In other words, I_2 can be made as small as we like provided T is large enough, and therefore I_2 exists. Hence L{f(t)} exists for $s > \alpha$.

Transform of some elementary functions :

Example : Find the Laplace transform of

(i) e^{at} (ii) cos at (iii) t^n

Solution :

(i)

$$L\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt$$

$$= \left(-\frac{e^{-(s-a)t}}{s-a}\right)_0^{\infty} = \frac{1}{s-a}, \quad s > a$$

(ii)

$$L\{\cos at\} = \int_0^{\infty} e^{-st} \cos at\,dt$$

$$= \text{Real}\left[\int_0^{\infty} e^{-st} e^{iat}\,dt\right]$$

$$= \text{Real}\left[L\{e^{iat}\}\right]$$

$$= \text{Real}\left[\frac{s+ia}{s^2+a^2}\right]$$

$$= \frac{s}{s^2+a^2}$$

(iii) Using the definition of Laplace transform of t^n, where n is a positive integer, we have

$$L\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

$$= \left(t^n \frac{e^{-st}}{-s} \right)_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt$$

$$= \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt$$

(integrating by parts)

Hence $L\{t^n\} = \frac{n}{s} L\{t^{n-1}\}$. Similarly, we can prove that

$$L\{t^{n-1}\} = \frac{n-1}{s} L\{t^{n-2}\}$$

$$L\{t^{n-2}\} = \frac{n-2}{s} L\{t^{n-3}\}$$

$$\cdots \qquad \cdots$$

$$L\{t^2\} = \frac{2}{s} L\{t\}$$

$$L\{t\} = \frac{1}{s^2}$$

Therefore $L\{t^n\} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \frac{1}{s^2} = \frac{n!}{s^{n+1}}$ which can be expressed in Gamma function as

$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$. Thus we can immediately generate the following table of transforms:

Table 1: Table of Laplace transforms

f(t)	f(s)	f(t)	f(s)
0	0	t^n	$\frac{n!}{s^{n+1}}$
1	$\frac{1}{s}$	sinat	$\frac{a}{s^2+a^2}$
e^{at}	$\frac{1}{s-a}$	cosat	$\frac{s}{s^2+a^2}$
e^{-at}	$\frac{1}{s+a}$	sinhat	$\frac{a}{s^2-a^2}$
t	$\frac{1}{s^2}$	coshat	$\frac{s}{s^2-a^2}$

Properties of Laplace transform :

We present a few important properties of Laplace transforms in the following theorems which will enable us to find the Laplace transform of a combination of functions whose transforms are known.

Theorem 2 : (Linearity property) If c_1 and c_2 are any two constants and if $F_1(s)$ and $F_2(s)$ are the Laplace transforms respectively of $f_1(t)$ and $f_2(t)$, then

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$
$$= c_1 F_1(s) + c_2 F_2(s)$$

Proof : (Left to the reader)

Theorem 3 : (Shifting property): If a function is multiplied by e^{at}, the transform of the resultant is obtained by replacing s by s-a in the transform of the original function. That is if $L\{f(t)\} = F(s)$, then $L\{e^{at} f(t)\} = F(s-a.)$

Proof : (left to reader)

Theorem 4 : (Multiplication by powers of t): If

$$L\{f(t)\} = F(s), \text{ then}$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$
$$= (-1)^n F^{(n)}(s)$$

where n = 1, 2, 3, ...

Proof : From the definition of Laplace transform

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Hence $$\frac{d}{ds} F(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right]$$

Interchanging the operation of differentiation and integration for which we assume that the necessary conditions are satisfied, and since there are two variables s and t, we use the notation of partial differentiation and obtain

$$\frac{d}{ds} F(s) = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} f(t)\} dt$$
$$= -\int_0^\infty e^{-st}\, t f(t)$$
$$= -L\{t f(t)\}$$

Therefore

$$L\{t f(t)\} = -\frac{d}{ds} F(s)$$

By repeated application of the above result, it can be shown that

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)$$

Theorem 5 (Differention property) : If $L\{f(t)\}=F(s)$ then

$$L\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Proof : From the definition of Laplace transform, we have

$$L\{f'(t)\}=\int_0^\infty e^{-st}f'(t)dt$$

$$=\left[e^{-st}f(t)\right]_0^\infty+s\int_0^\infty e^{-st}f(t)dt$$

(integrating by parts)

$$=-f(0)+s\int_0^\infty e^{-st}f(t)dt$$

$$=-f(0)+sL\{f(t)\}$$

$$=sF(s)-f(0)$$

Similarly, it can be shown that

$$L\{f''(t)\}=sL\{f'(t)\}-f'(0)$$

$$=s\{sF(s)-f(0)\}-f'(0)$$

$$=s^2F(s)-sf(0)-f'(0)$$

$$L\{f'''(t)\}=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$$

Thus, in general,

$$L\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

This property is very useful for solving differential equation.

Example : Find the Laplace transform of

(i) $e^{at}\cos bt$ (ii) $e^{at}t^n$ (iii) t^2e^{at}

Solution :

(i) Using the shifting property

$$L\{e^{at}\cos bt\}=\left.\frac{s}{s^2+b^2}\right|_{s\to(s-a)}=\frac{s-a}{(s-a)^2+b^2}$$

(ii) Using the shifting property

$$L\{e^{at}t^n\}=\left.\frac{n!}{s^{n+1}}\right|_{s\to(s-a)}=\frac{n!}{(s-a)^{n+1}}$$

(iii) Using the differentiation property

$$L\{t^2e^{at}\}=(-1)^2\frac{d^2}{ds^2}L\{e^{at}\}=\frac{d^2}{ds^2}\frac{1}{s-a}=\frac{2}{(s-a)^3}$$

Alternatively using the shifting property

$$L\{e^{at}t^2\}=\left.\frac{2!}{s^3}\right|_{s\to(s-a)}=\frac{2!}{(s-a)^3}$$

Theorem 6 (Initial value theorem) : If f(t) and $f'(t)$ are Laplace transformable and F(s) is the Laplace transform of f(t), then the behaviour of f(t) in the neighbourhood of t=0 corresponds to the behaviour of F(s) in the neighbourhood of $s=\infty$.

Mathematically, $\underset{t\to 0}{\text{Lt}}\, f(t) = \underset{s\to\infty}{\text{Lt}}\, sF(s)$

Proof:

From the property of derivative, we have

$$L\{f'(t)\} = sF(s) - f(0)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

Taking the limit as $s \to \infty$ on both sides, we get

$$\underset{s\to\infty}{\text{Lt}} \int_0^\infty e^{-st} f'(t)\,dt = \underset{s\to\infty}{\text{Lt}}\, sf(s) - \underset{s\to\infty}{\text{Lt}}\, f(0)$$

Since s is independent of t, we can take the limit before integrating the left hand side of the above equation, thus getting

$$\underset{s\to\infty}{\text{Lt}} \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty \left[\underset{s\to\infty}{\text{Lt}}\, e^{-st} f'(t)\right] dt = 0$$

and thus

$$\underset{s\to\infty}{\text{Lt}}\, sF(s) - \underset{s\to\infty}{\text{Lt}}\, f(0) = 0$$

$$\Rightarrow \underset{s\to\infty}{\text{Lt}}\, sF(s) = f(0) = \underset{t\to 0}{\text{Lt}}\, f(t)$$

Hence the result. For example, let f(t) be a polynomial of degree n of the form

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

Its Laplace transform is

$$F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \cdots + \frac{n!a_n}{s^{n+1}}$$

$$\Rightarrow sF(s) = a_0 + \frac{a_1}{s} + \frac{2a_2}{s^2} + \cdots + \frac{n!a_n}{s^n}$$

$$\therefore \underset{s\to\infty}{\text{Lt}}\, sF(s) = a_0 = f(0)$$

Example : Prove that $L\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$

Solution : Let $f(t) = \int_0^t \frac{\sin u}{u}\,du$ then $f(0) = 0,\ f'(t) = \frac{\sin t}{t}$

giving

$$tf'(t) = \sin t$$

$$\Rightarrow L\{tf'(t)\} = \frac{1}{s^2+1}$$

$$\Rightarrow -\frac{d}{ds} L\{f'(t)\} = \frac{1}{s^2+1}$$

$$\Rightarrow -\frac{d}{ds}\{sF(s) - f(0)\} = \frac{1}{s^2+1}$$

$$\Rightarrow \frac{d}{ds}\{sF(s)\} = -\frac{1}{s^2+1}$$

Integrating $\quad sF(s) = -\tan^{-1} s + c$

By initial value theorem

$$\underset{s\to\infty}{Lt}\, sF(s) = \underset{t\to 0}{Lt}\, f(t) = 0$$

Therefore

$$\underset{s\to\infty}{Lt}\, sF(s) = 0 = -\underset{s\to\infty}{Lt}\left(\tan^{-1}\right) + C$$

givin $\quad C = \frac{\pi}{2}$

Hence $\quad sF(s) = \frac{\pi}{2} - \tan^{-1} = \tan^{-1}\frac{1}{s}$

$$\Rightarrow F(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Theorem 7 (Final value theorem) : If $f(t)$ and $f'(t)$ are Laplace transformable and F(s) is the Laplace transform of f(t), then the behaviour of f(t) in the neighbourhood of $t = \infty$ corresponds to the behaviour of sF(s) in the neighbourhood of $s = 0$. Mathematically $\quad \underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$

Proof : From the property of derivative, we have

$$L\{f'(t)\} = sF(s) - f(0)$$

i.e. $\quad \int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$

Taking the limit as $s \to 0$ on both sides of the above equation, we have

$$\underset{s\to 0}{Lt}\int_0^\infty e^{-st} f'(t)\,dt = \underset{s\to 0}{Lt}\, sF(s) - \underset{s\to 0}{Lt}\, f(0)$$

But $\quad \int_0^\infty \underset{s\to 0}{Lt}\, e^{-st} f'(t)\,dt = \int_0^\infty f'(t)\,dt$

$$= [f(t)]_0^\infty$$

$$= \underset{t\to\infty}{Lt}\, f(t) - f(0)$$

Using this result in the above equation, we get

$$\operatorname*{Lt}_{s\to 0} sF(s) - \operatorname*{Lt}_{s\to 0} f(0) = \operatorname*{Lt}_{t\to\infty} f(t) - f(0)$$

$$\Rightarrow \operatorname*{Lt}_{s\to 0} sF(s) = \operatorname*{Lt}_{t\to\infty} f(t)$$

Example :

Prove that $L\left\{\int_t^\infty \frac{\cos u}{u} du\right\} = \frac{\ln(s^2+1)}{2s}$

Proof : Let $f(t) = \int_t^\infty \frac{\cos u}{u} du$

giving

$$f'(t) = -\frac{\cos t}{t}$$

$$\Rightarrow t f'(t) = -\cos t$$

$$\Rightarrow L\{t f'(t)\} = -L\{\cos t\}$$

$$\Rightarrow -\frac{d}{ds}\{sF(s) - f(0)\} = -\frac{s}{s^2+1}$$

$$\Rightarrow \frac{d}{ds}\{sF(s)\} = \frac{s}{s^2+1}$$

Integrating $sF(s) = \frac{1}{2}\ln(s^2+1) + c$

By final value theorem

$$\operatorname*{Lt}_{s\to 0} sF(s) = \operatorname*{Lt}_{t\to\infty} f(t) = 0$$

so that

$$\operatorname*{Lt}_{s\to 0} sF(s) = 0 = \operatorname*{Lt}_{s\to 0} \frac{1}{2}\ln(s^2+1) + c \quad \text{giving } c=0$$

Hence,

$$sF(s) = \frac{1}{2}\ln(s^2+1)$$

$$\therefore F(s) = \frac{\ln(s^2+1)}{2s}$$

Theorem 8 (Division by t) : If $L\{f(t)\} = F(s)$ then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$$

Proof : From the definition of Laplace transform

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Integrating the above equation with respect to s between the limits s to ∞, we get

$$\int_s^\infty F(s)\,ds = \int_s^\infty\left[\int_0^\infty e^{-st} f(t)\,dt\right]ds$$

$$= \int_0^\infty f(t)\left(\int_s^\infty e^{-st} ds\right)dt \quad \text{(by changing the order of integration)}$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_0^\infty dt$$

$$= \int_0^\infty \frac{f(t)}{t} e^{-st} dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

Note : In applying this rule, one should be careful. Since $\frac{f(t)}{t}$ may have an infinite discontinuity at $t = 0$, it may not be integrable. If $\frac{f(t)}{t}$ is not integrable, then its Laplace transform does not exists. For example, at $t = 0$, the function $\frac{\sin t}{t}$ does not have an infinite discontinuity, while the function $\frac{\cos t}{t}$ has an infinite discontinuity.

Example : Find the Laplace Transform of

(i) $\frac{\sin at}{t}$ (ii) $\frac{1-\cos t}{t}$

Solution :

(i) We know $L\{\sin at\} = \frac{a}{s^2+a^2}$. Using theorem (8), we have

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2} ds = a.\frac{1}{a}\tan^{-1}\frac{s}{a}\bigg|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \tan^{-1}\frac{a}{s}$$

(ii) We know $L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$. Using theorem (8), we have

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty\left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds$$

$$= \left[\ln s - \frac{1}{2}\ln(s^2+1)\right]_s^\infty$$

$$= \left[\ln\frac{s}{(s^2+1)^{\frac{1}{2}}}\right]_s^\infty$$

$$=\left[\ln\frac{1}{\left(1+\frac{1}{s^{2}}\right)^{\frac{1}{2}}}\right]_{s}^{\infty}=0-\ln\frac{s}{\left(s^{2}+1\right)^{\frac{1}{2}}}=\ln\frac{\left(s^{2}+1\right)^{\frac{1}{2}}}{s}$$

Example : Find the Laplace transform of (i) $\sin\left(\sqrt{t}\right)$ (ii) $\frac{\cos\sqrt{t}}{\sqrt{t}}$

Solution :

(i) Using series

$$\sin\sqrt{t}=\sqrt{t}-\frac{\left(\sqrt{t}\right)^{3}}{3!}+\frac{\left(\sqrt{t}\right)^{5}}{5!}-\frac{\left(\sqrt{t}\right)^{7}}{7!}+\cdots$$

$$=t^{\frac{1}{2}}-\frac{t^{\frac{3}{2}}}{3!}+\frac{t^{\frac{5}{2}}}{5!}-\frac{t^{\frac{7}{2}}}{7!}+\cdots$$

Then the Laplace transform is

$$L\left\{\sin\sqrt{t}\right\}=\frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}-\frac{\Gamma\left(\frac{5}{2}\right)}{3!s^{\frac{5}{2}}}+\frac{\Gamma\left(\frac{7}{2}\right)}{5!s^{\frac{7}{2}}}-\frac{\Gamma\left(\frac{9}{2}\right)}{9!s^{\frac{9}{2}}}+\cdots$$

$$=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\left\{1-\frac{1}{2^{2}s}+\frac{\left(\frac{1}{2^{2}s}\right)^{2}}{2!}-\frac{\left(\frac{1}{2^{2}s}\right)^{3}}{3!}+\cdots\right\}$$

$$=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{2^{2}s}}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\frac{1}{4s}}$$

(ii) Let $f(t)=\sin\sqrt{t}$. Then $f'(t)=\frac{\cos\sqrt{t}}{2\sqrt{t}}, f(0)=0$. Hence

$$L\{f'(t)\}=\frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$$

$$\Rightarrow\frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}=sF(s)-f(0)=\frac{\sqrt{\pi}}{2s^{\frac{1}{2}}}e^{-\frac{1}{4s}},\text{[Here } F(s)=L\{f(t)\}\text{]}$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}=\frac{\sqrt{\pi}}{s^{\frac{1}{2}}}e^{-\frac{1}{4s}}\quad\text{(using theorem 5)}$$

Transform of Bessel's function :

Bessel's function arises in several problems involving circular and cylindrical geometry. It is therefore useful to find the Laplace transform of Bessel functions of the first kind.

Exercise : Find the Laplace transform of (i) $J_0(t)$ (ii) $t\,J_0(t)$

Solution :

(i) From the definition of Bessel function, we have

$$J_n(t)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\overline{|n+r+1}}\left(\frac{t}{2}\right)^{n+2r}$$

For $n=0$, we have

$$J_0(t)=1-\frac{t^2}{2^2}+\frac{t^4}{2^2\times4^2}-\frac{t^6}{2^2\times4^2\times6^2}+\cdots$$

Thus

$$L\{J_0(t)\}=\frac{1}{s}-\frac{2!}{2^2s^3}+\frac{4!}{2^2\times4^2\times s^5}-\frac{6!}{2^2\times4^2\times6^2\times s^7}+\cdots$$

$$=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1\times3}{2\times4}\left(\frac{1}{s^2}\right)^2-\frac{1\times3\times5}{2\times4\times6}\left(\frac{1}{s^2}\right)^3+\cdots\right]$$

$$=\frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-\frac{1}{2}}=\frac{1}{\sqrt{1+s^2}}$$

(ii) $$L\{tJ_0(t)\}=-\frac{d}{ds}L\{J_0(t)\}=-\frac{d}{ds}\left(\frac{1}{\sqrt{1+s^2}}\right)=\frac{s}{(1+s^2)^{\frac{3}{2}}}$$

Transform of dirac delta function :

According to the notion in mechanics, we come across a very large force (ideally infinite) acting for a short distance (ideally zero time) known as impulsive force. Thus we have a function which is non-zero in a very short interval. The dirac delta function may be thought of as a generalisation of this concept.

Consider the function having the following property:

$$\delta_\varepsilon(t)=\begin{cases}\frac{1}{2}\varepsilon, & |t|<\varepsilon\\ 0, & |t|>\varepsilon\end{cases}$$

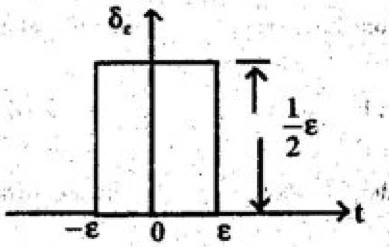
Thus $\displaystyle\int_{-\infty}^{\infty}\delta_\varepsilon(t)\,dt=\int_{-\varepsilon}^{\varepsilon}\frac{1}{2\varepsilon}dt=1$

Let $f(t)$ be any function which is integrable in the interval $(-\varepsilon,\varepsilon)$. Then using the mean-value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty}f(t)\delta_\varepsilon(t)\,dt=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}f(t)\,dt=f(\xi),\quad -\varepsilon<\xi<\varepsilon$$

Thus we may regard $\delta(t)$ as a limiting function approached by $\delta_\varepsilon(t)$ as $\varepsilon \to 0$, i.e.

$$\delta_\varepsilon(t) = \underset{\varepsilon\to 0}{Lt}\, \delta_\varepsilon(t)$$



As $\varepsilon \to 0$, we have, the relation

$$\delta_\varepsilon(t) = \underset{\varepsilon\to 0}{Lt}\, \delta_\varepsilon(t) = \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

This limiting function $\delta(t)$ thus defined is known as Dirac delta function or the unit impulse function. Its profile is depicted. Dirac originally called it an improper function as there is no proper function with these properties. In fact we can observe that

$$1 = \int_{-\infty}^{\infty} \delta(t)\,dt = \underset{\varepsilon\to 0}{Lt} \int_{|t|<\varepsilon} \delta_\varepsilon(t)\,dt = \underset{\varepsilon\to 0}{Lt}\, 0 = 0$$

Obviously, this contradiction implies that $\delta(t)$ cannot be a function in the ordinary sense.

One useful property of Dirac delta function :

Let f(t) be any continuous function. Then

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$$

Proof :

Consider the function

$$\delta_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & a < t < a+\varepsilon \\ 0 & \text{elsewhere} \end{cases}$$

Using mean-value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t-a) f(t)\,dt = \frac{1}{\varepsilon}\int_{a}^{a+\varepsilon} f(t)\,dt = f(a+\theta\varepsilon),\ 0<\theta<1$$

Now, Taking the limit as $\varepsilon \to 0$, we obtain

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$$

Example : Find the Laplace transform of Dirac delta function.

Solution : From the property of Direct delta function, we have

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$$

In particular, if $f(t) = e^{-st}$, then

$$L\{\delta(t-a)\}=\int_0^\infty e^{-st}\delta(t-a)dt=e^{-as},\ a>0.$$

Transform of a periodic function :

A function f(t) is called periodic with period T, if f(t + T)=f(t) for all values of t and T>0. For example, the trigonometric function sint and cost are periodic function of period 2π.

Theorem 9 : If f(t) is a periodic function with period T, then

$$L\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

Proof : From the definition of Laplace transform, we have

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^T e^{-st}f(t)dt+\int_T^\infty e^{-st}f(t)dt$$

If we substitute t = u + T in the second integral on the right hand side and write dt = du, we get

$$\begin{aligned}L\{f(t)\}&=\int_0^T e^{-st}f(t)dt+\int_0^\infty e^{-s(u+T)}f(u+T)du\\&=\int_0^T e^{-st}f(t)dt+e^{-sT}\int_0^\infty e^{-su}f(u)du\\&=\int_0^T e^{-st}f(t)dt+e^{-sT}L\{f(t)\}\end{aligned}$$

Rearranging, we get $L\{f(t)\}=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$

Example : Find the Laplace transform of

$$f(t)=\begin{cases}1, & 0\le t\le 2\\-1, & 2\le t\le 4\end{cases}$$

$$f(t+4)=f(t)$$

Solution : In this problem, f(t) is a periodic function of period 4; we therefore have

$$\begin{aligned}L\{f(t)\}&=\frac{1}{1-e^{-4s}}\int_0^4 e^{-st}f(t)\,dt\\&=\frac{1}{1-e^{-4s}}\left[\int_0^2 e^{-st}f(t)\,dt+\int_2^4 e^{-st}f(t)\,dt\right]\\&=\frac{1}{1-e^{-4s}}\left[\int_0^2 e^{-st}.1\,dt+\int_2^4 e^{-st}(-1)\,dt\right]\\&=\frac{1}{1-e^{-4s}}\left[\frac{-2e^{-2s}}{s}+\frac{e^{-4s}}{s}+\frac{1}{s}\right]\end{aligned}$$

So far we have discussed various properties of the Laplace transform and studied the Laplace transform of some simple functions. However, if the Laplace transform technique is to be useful in applications, we have to consider the reverse problem too, i.e. we have to find the original function f(t)when we know its Laplace transform F(s). Thus if

$$L\{f(t)\} = F(s)$$

then $f(f) = L^{-1}\{F(s)\}$.

In other words, the inverse Laplace transform of a given function F(s) is that function f(t) whose Laplace transform is F(s). It can be established that f(t) is unique. Here L^{-1} is known as inverse Laplace transform operator. From the elementary definition and from the result obtained, thus in finding the inverse Laplace transform which are Laplace transform of some elementary functions as in the following table:

Table 2 Table of inverse Laplace transform

F(s)	f(t)	F(s)	f(t)	F(s)	f(t)
0	0	$\frac{a}{S^2-a^2}$	sinh at	$\frac{a}{S^2+a^2}$	sin at
$\frac{1}{S}$	1	$\frac{S}{S^2-a^2}$	cosh at	$\frac{S}{S^2+a^2}$	cosat
$\frac{1}{S-a}$	e^{at}	$\frac{1}{\sqrt{1+S^2}}$	$J_0(t)$	$\frac{1}{(S-a)^3}$	$\frac{t^2e^{at}}{2}$
$\frac{1}{S+a}$	e^{-at}	$\frac{S}{(1+S^2)^{\frac{3}{2}}}$	$t\,J_0(t)$	$\frac{1}{S^n+1}$	t^n
$\frac{1}{S^2}$	t	$\frac{e^{-\frac{1}{4S}}}{S^{\frac{3}{2}}}$	$\frac{2}{\sqrt{\pi}}\sin\sqrt{t}$	$\frac{1}{S}\tan^{-1}\frac{1}{S}$	sin at

In most of the problems we have considered earlier, $L\{f(t)\}$ is a simple rational function. The linearity property holds true even in the case of inverse transform. That is, if $F_1(s)$ and $F_2(s)$ are the Laplace transform of $f_1(t)$ and $f_2(t)$, and C_1 and C_2 are any two constants, then

$$L^{-1}\{C_1F_1(s)+C_2F_2(s)\} = C_1L^{-1}\{F_1(s)\}+C_2L^{-1}\{F_2(s)\}$$

By expressing $L\{f(t)\}$ as partial functions, we should be able to recognise them as the transform of some known function; with the help of which we can write down the inverse transform. Similarly, shifting property is also useful in constructing the inverse transform of some functions, which is stated in the following theorem:

Theorem 10 : If $L\{f(t)\}=F(s)$ then

$$L^{-1}F(s+a)=e^{-at}L^{-1}\{F(s)\}$$

(proof is left to reader)

Example : Obtain the inverse Laplace transform of

$$\frac{4s^2-3s+5}{(s+1)(s^2-2s+2)}.$$

Solution : Using partial fraction expression, we can write

$$\frac{4s^2-3s+5}{(s+1)(s^2-2s+2)}=\frac{A}{s+1}+\frac{Bs+C}{s^2-2s+2}$$

whence we can find that

$$A=\frac{12}{5},\ B=\frac{8}{5},\ C=\frac{1}{5}$$

The given expression can now be written as

$$\frac{4s^2-3s+5}{(s+1)(s^2-2s+2)}=\frac{12}{5}\frac{1}{s+1}+\frac{8}{5}\frac{s}{(s-1)^2+1^2}+\frac{1}{5}\frac{1}{(s-1)^2+1^2}$$

Thus,

$$L^{-1}\left\{\frac{4s^2-3s+5}{(s+1)(s^2-2s+2)}\right\}=\frac{12}{5}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{8}{5}L^{-1}\left\{\frac{(s-1)+1}{(s-1)^2+1^2}\right\}+$$

$$\frac{1}{5}L^{-1}\left\{\frac{1}{(s-1)^2+1^2}\right\}$$

$$=\frac{12}{5}e^{-t}+\frac{8}{5}e^{t}L^{-1}\left\{\frac{s+1}{s^2+1}\right\}+\frac{1}{5}e^{t}L^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

(by using shifting property)

$$=\frac{12}{5}e^{-t}+\frac{8}{5}e^{t}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\}+L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]+\frac{1}{5}e^{t}\sin t$$

$$=\frac{12}{5}e^{-t}+\frac{8}{5}e^{t}[\cos t+\sin t]+\frac{1}{5}e^{t}\sin t$$

Theorem 11 : If f(t) is a piecewise continuous function and satisfies the condition of exponential order α such that $\underset{t\to 0}{Lt}\frac{f(t)}{t}$ exists, then for $s>\alpha$

$$L^{-1}\left\{\frac{F(s)}{s}\right\}=\int_0^t f(x)\,dx$$

Proof : Let

$$g(t) = \int_0^t f(x)dx, \text{ then}$$

$g(0) = 0$ and $g'(t) = f(t)$. Also

$$L\{g'(t)\} = sL\{g(t)\} - g(0) = sL\{g(t)\}$$

$$\Rightarrow \quad L\{f(t)\} = sL\{g(t)\}$$

Therefore $L\{g(t)\} = \dfrac{F(s)}{s}$

$$\text{Hence } L^{-1}\left\{\frac{F(s)}{s}\right\} = g(t) = \int_0^t f(x)\,dx$$

This result can be generalized to show that

$$L^{-1}\left\{\frac{F(s)}{s^n}\right\} = \int \int \cdots \int_0^t f(t)\,dt^n$$

Theorem 12: (change of scale property): If

$$L^{-1}\{F(s)\} = f(t) \text{ then } L^{-1}\{F(s)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

(proof is left to the reader)

Convolution theorem (Faltung theorem) :

We often come across functions which are not transform of some known function, but then, they can possibly be expressed as a product of two functions, each of which is the transform of a known function. Thus we may be able to write the given function as F(s)G(s), where F(s)and G(s) are known to be transform of the function f(t) and g(t), respectively.

Theorem 13 : If F(s) and G(s) are the Laplace transform of f(t) and g(t) respectively, then F(s) and G(s) is the Laplace transform of

$$\int_0^t f(t-u)\,g(u)\,du$$

$$\text{i.e. } L^{-1}\{F(s)G(s)\} = \int_0^t f(t-u)\,g(u)\,du$$

This integral is called the convolution of f and g and is denoted by the symbol $f*g$.

Proof : From the definition of Laplace transform, we have

$$F(s)G(s) = \left[\int_0^\infty e^{-sv} f(v)\,dv\right]\left[\int_0^\infty e^{-su} g(u)\,du\right]$$

$$= \int_0^\infty\int_0^\infty e^{-s(u+v)} f(v)\,g(u)\,dv\,du$$

$$= \int_0^\infty g(u)\left\{\int_0^\infty e^{-s(v+u)}\, f(v)\,dv\right\}du$$

Let $u + v = t$ in the inner integral. Then

$$F(s)G(s) = \int_0^\infty g(u)\left\{\int_u^\infty e^{-st} f(t-u)\,dt\right\}du$$

Change the order of integration is shown in the figure. Then we get

$$F(s)G(s) = \int_0^\infty \left\{\int_0^t e^{-st} f(t-u)\, g(u)\,du\right\}dt$$

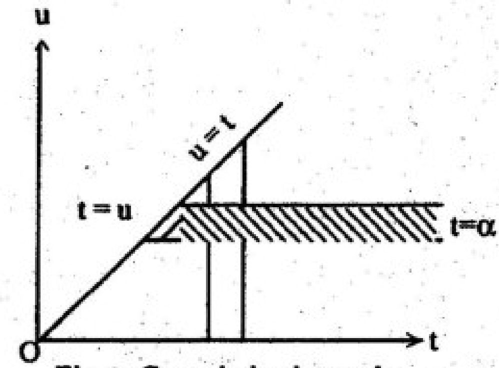


Figure Convolution integral

$$= \int_0^\infty e^{-st}\left\{\int_0^t f(t-u)\, g(u)\,du\right\}dt$$

$$= L\left[\int_0^t f(t-u)\,g(u)\,du\right]$$

hence the result.

Definition : We define the Laplace convolution of f(t) and g(t) by the integral

$$f * g = \int_0^t f(t-u)\,g(u)\,du$$

It can be verified that f and g can be interchanged in the convolution, i.e. f and g are commutative.Thus,

$$f * g = \int_0^t f(t-u)\,g(u)\,du = \int_0^t g(t-u)\,f(u)\,du = g * f$$

Example : Find the Laplace inverse of

(i) $\dfrac{s}{(s^2+a^2)^2}$ (ii) $\dfrac{1}{(s^2+a^2)^2}$ (iii) $\dfrac{1}{s^2(s^2+1)^2}$

Solution :

(i) Method 1.

Let $F(s) = \dfrac{1}{s^2+a^2}$, then $f(t) = \dfrac{1}{a}\sin at$

Now, $\dfrac{d}{ds}F(s) = -\dfrac{2s}{(s^2+a^2)}$. Therefore,

$$L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -2L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow (-1)tf(t) = -2L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2}f(t) = \frac{t}{2}.\frac{1}{a}\sin at$$

Method 2 :

We can write

$$\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2}.\frac{1}{s^2+a^2} = F(s).G(s)$$

Then $f(t) = \cos at$, $g(t) = \frac{1}{a}\sin at$

We have by convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\{F(s).G(s)\} = \int_0^t \cos au \frac{\sin a(t-u)}{a}du$$

$$= \frac{1}{a}\int_0^t (\sin at\cos au - \cos at\sin au)\cos au\, du$$

$$= \frac{1}{a}\sin at\int_0^t \cos^2 au\, du - \frac{\cos at}{a}\int_0^t \sin au\cos au\, du$$

$$= \frac{1}{a}\sin at\int_0^t \frac{1+\cos 2au}{2}du - \frac{\cos at}{2a}\int_0^t \sin 2au\, du$$

$$= \frac{1}{a}\sin at\left(\frac{t}{2}+\frac{\sin 2at}{4a}\right) - \frac{\cos at}{4a}(1-\cos 2at)$$

$$= \frac{t\sin at}{2a}$$

(ii) Solution :

Method 1:

We have from (i) above that

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin at}{2a} = f(t) \text{ (say)}$$

Therefore

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s}.\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \frac{u\sin au}{2a}du \qquad \text{(using theorem 11)}$$

$$= \frac{1}{2a}\left\{\left[-\frac{u\cos au}{a}\right]_0^t + \left[\frac{1}{a^2}\sin au\right]_0^t\right\} \qquad \text{(integrating by parts)}$$

$$= \frac{1}{2a}\left[-\frac{1}{a}t\cos at + \frac{\sin at}{a^2}\right]$$

$$= \frac{1}{2a^3}[\sin at - at\cos at]$$

Method 2 :

We can write

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2} = F(S).G(S)$$

then $f(t) = \frac{1}{a}\sin at,\ g(t) = \frac{1}{a}\sin at$

Now using the convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \int_0^t \frac{1}{a}\sin au\frac{1}{a}\sin a(t-u)du$$

$$= \frac{1}{a^2}\int_0^t \sin au(\sin at\cos au - \cos at\sin au)\,du$$

$$= \frac{\sin at}{a^2}\int_0^t \sin au\cos au\,du - \frac{\cos at}{a^2}\int_0^t \sin^2 au\,du$$

$$= \frac{\sin at}{2a^2}\int_0^t \sin 2au\,du - \frac{\cos at}{a^2}\int_0^t \frac{(1-\cos 2au)}{2}du$$

$$= \frac{\sin at}{2a^2}\left(\frac{1}{2a} - \frac{\cos at}{2a}\right) - \frac{\cos at}{a^2}\left(\frac{t}{2} - \frac{\sin 2at}{4a}\right)$$

$$= \frac{1}{2a^3}(\sin at - at\cos at) \quad \text{(simplifying)}$$

(iii) Method 1 : Let $F(s) = \frac{1}{(s+1)^2}$ then $f(t) = e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} = te^{-t}$

Now, $L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s+1)^2}\right\} = \int_0^t ue^{-u}du = -e^{-t} - te^{-t} + 1$ (integrating by parts)

Again

$$L^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\right\} = L^{-1}\left[\frac{1}{s}\left\{\frac{1}{s}\frac{1}{(s+1)^2}\right\}\right]$$

$$= \int_0^t\left(-e^{-u} - ue^{-u} + 1\right)du$$

$$= -1 + e^{-t} - (-e^{-t} - te^{-t} + 1) + t$$

$$= te^{-t} + 2e^{-t} - 2 + t$$

Method 2 : We can write $\frac{1}{s^2(s+1)^2} = \frac{1}{s^2}.\frac{1}{(s+1)^2} = F(s).G(s)$

then $f(t) = t, g(t) = e^{-t}t$. Using convolution theorem

$$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t ue^{-u}(t-u)\,du$$

$$= \int_0^t (ut - u^2)e^{-u}\,du$$

$$= te^{-t} + 2e^{-t} + t - 2$$

Example : Find inverse Laplace transform of

(i) $\ln\left(1+\frac{1}{s^2}\right)$ (ii) $\ln\left(\frac{s+2}{s+1}\right)$

Solution : (i) Let $F(s)= \ln\left(1+\frac{1}{s^2}\right) = \ln\frac{s^2+1}{s^2}$

then $\frac{d}{ds}F(s) = \frac{s^2}{s^2+1}\frac{s^2\,2s-(s^2+1)\,2s}{s^4} = \frac{2s^3-2s^3-2s}{s^2(s^2+1)} = \frac{2}{s(s^2+1)}$

$$L^{-1}\left\{\frac{d}{ds}F(s)\right\} = -L^{-1}\left\{\frac{2}{s(s^2+1)}\right\}$$

$$\Rightarrow (-1)t\,f(t) = -2L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$$

$$\therefore f(t) = L^{-1}\{F(s)\} = \frac{2}{t}L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$$

Since $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

Therefore,

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u\,du = 1-\cos t$$

$$\therefore f(t) = \frac{2(1-\cos t)}{t}$$

(ii)

Let $F(s) = \ln\frac{s+2}{s+1}$

$$\therefore \frac{d}{ds}F(s) = \frac{s+1}{s+2}\frac{(s+1)-(s+2)}{(s+1)^2}$$

$$= \frac{-1}{(s+2)(s+1)}$$

Therefore

$$L^{-1}\left[\frac{d}{ds}F(s)\right] = -L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

$$\Rightarrow (-1)tf(t) = -L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

$$\Rightarrow f(t) = L^{-1}\{F(s)\} = \frac{1}{t}L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

By partial fraction we can write that

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Therefore

$$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t} - e^{-2t}$$

$$\therefore f(t) = \frac{e^{-t} - e^{-2t}}{t}$$

Solution of Ordinary differential equation

The Laplace transform technique is one of the powerful tools for solving physical problems involving ordinary differential equation (ODE), particularly initial value problems. It reduces the solution of ODE to the solution of an algebraic equation. This method has a particular advantage in finding the solution of an initial value problem, without first finding the general solution and then using the given initial conditions for evaluating the arbitrary constants.

Example: Solve $\ddot{x} + 4\dot{x} = -8t$; $x(0) = x'(0) = 0$

Solution: Taking Laplace transform of the given ODE, we get

$$s^2X - sx(0) - x'(0) + 4\{sX - x(0)\} = -\frac{8}{s^2}$$

(here $L\{x\}=X$)

Using the initial conditions we get

$$s^2X + 4sX = -\frac{8}{s^2}$$

therefore $X = -\frac{8}{s^3(s+1)}$

and so $x = L^{-1}\{X\} = -8L^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$

Now let us find the $L^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$

$$\text{Since } L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2},\ L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

Therefore using convolution theorem

$$L^{-1}\left\{\frac{1}{s^3(s+1)}\right\} = \int_0^t e^{-(t-u)}\frac{u^2}{2}du$$

$$= \frac{e^{-t}}{2}\int_0^t u^2e^u du$$

$$= \frac{e^{-t}}{2}\left[u^2e^u - 2ue^u + 2e^u\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[t^2e^t - 2te^t + 2e^t - 2\right]$$

$$= \frac{1}{2}\left[t^2 - 2t + 2 - 2e^{-t}\right]$$

Hence the solution.

Example. Solve

$$ty'' + y' + ty = 0,\quad y(0) = 1,\ y'(0) = 0$$

Solution: We have

$$L\{ty''\} + L\{y'\} + L\{ty\} = 0$$

$$-\frac{d}{ds}L\{y''\} + \{sY - y(0)\} - \frac{d}{ds}L\{y\} = 0$$

$$\Rightarrow -\frac{d}{ds}\left\{s^2Y - sy(0) - y'(0)\right\} + sY - 1 - \frac{dY}{ds} = 0$$

Here $L\{y\}=Y$

Applying initial conditions

$$-\frac{d}{ds}\left\{s^2Y - s\right\} + \{sY - 1\} - \frac{dY}{ds} = 0$$

$$\Rightarrow (s^2+1)\frac{dY}{ds} + sY = 0$$

Which is a first order O.D.E. Rewriting we get

$$\frac{dY}{Y} + \frac{sds}{s^2+1} = 0$$

Integrating

$$\ln Y + \frac{1}{2}\ln(s^2+1) = \ln c$$

$$Y = \frac{c}{\sqrt{s^2+1}}$$

Taking inverse Laplace transform we get $y = cJ_0(t)$, where $J_0(t)$ is a Bessel function of order zero.

Since $y(0)=1=cJ_0(0)=c$, the required solution is $y=J_0(t)$

Example Solve $\frac{dx}{dt} = 2x - 3y$

$$\frac{dy}{dt} = y - 2x; \quad x(0) = 8,\ y(0) = 3$$

Solution We have by taking Laplace transform

$$sX - x(0) = 2X - 3Y \quad \text{or } (s-2)X + 3Y = 8$$

$$sY - y(0) = Y - 2X \quad \text{or } 2X + (s-1)Y = 3$$

Solving for X and Y we get (by cramer's rule)

$$X = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$Y = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

Then $x = L^{-1}\{X\} = 5e^{-t} + 3e^{4t}$ $\quad y = L^{-1}\{Y\} = 5e^{-t} - 2e^{4t}$

Solution of partial differential equation:

A large number of problems in science and engineering involve the solution of linear partial differential equation. A function of two or more variables may also have a Laplace transform. Suppose x and t are two independent variables, consider t as the principal variable and x as the secondary variable. When the Laplace transform is applied with t as a variable, the PDF is reduced to an ordinary differential equation of the t-transform V(x,s), where xis the independent variable. The general solution V(x,s) of the ODE is then fitted to the BCs of the original problem. Finally the solution u(x,t) is obtained by finding the inverse Laplace transform. Thus the Laplace transform is specially suited to solving initial boundary value problem (IBVP) when conditions are prescribed at t=0.

Example If u(x,t) is a function of two variables x and t, prove that

(i) $L\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - u(x,0)$

(ii) $L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2U(x,s) - su(x,0) - u_t(x,0)$

(iii) $L\left\{\frac{\partial u}{\partial x}\right\} = \frac{dU(x,s)}{dx}$

(iv) $L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2U(x,s)}{d^2x}$

where $U(x,s) = L\{u(x,t)\}$

Proof:

(i) $$L\left\{\frac{\partial u}{\partial t}\right\} = \int_0^{\infty} e^{-st}\frac{\partial u}{\partial t}dt = \underset{p\to\infty}{Lt}\int_0^{p} e^{-st}\frac{\partial u}{\partial t}dt$$

$$= \underset{p\to\infty}{Lt}\left[\left\{e^{-st}u(x,t)\right\}_0^p + s\int_0^p e^{-st}u(x,t)\,dt\right]$$

$$= -u(x,0) + s\int_0^{\infty} e^{-st}u(x,t)\,dt = sL\{u(x,t)\} - u(x,0)$$

$$= sU(x,s) - u(x,0)$$

(ii) $$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = L\left[\frac{\partial v}{\partial t}\right],\ V = \frac{\partial u}{\partial t}$$

$$= sL\{V(x,t)\} - V(x,0) \qquad \text{(using (i))}$$

$$= s\{sV(x,s)-u(x,0)\} - u_t(x,0) = s^2V(x,s) - su(x,0) - u_t(x,0)$$

(iii) $$L\left\{\frac{\partial u}{\partial x}\right\} = \int_0^{\infty} e^{-st}\frac{\partial u}{\partial x}dt$$

$$= \frac{d}{dx}\int_0^{\infty} e^{-st}u(x,t)dt$$

$$= \frac{d}{dx}L\{u(x,t)\} = \frac{dU(x,s)}{dx}$$

(iv) $$L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = L\left[\frac{\partial \bar{u}}{\partial x}\right] = \frac{d}{dx}\{\bar{u}(x,t)\},\ \bar{u} = \frac{\partial u}{\partial x}$$

$$= \frac{d}{dx}\left[L\left\{\frac{\partial u(x,t)}{\partial x}\right\}\right]$$

$$= \frac{d}{dx}\left[\frac{d}{dx}L\{u(x,t)\}\right]$$

$$= \frac{d^2}{dx^2}U$$

Example Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2};\quad u(0,t) = 1,\ u(x,0) = 0$$

Solution Taking the Laplace transform of the given partial differential equation and the condition $u(0,t)=1$, we find

$$sU - u(x,0) = \frac{d^2U}{dx^2}$$

$$\text{or } \frac{d^2U}{dx^2} - sU = 0,\ U(x,s) = L\{u(x,t)\}$$

$$\text{and } U(0,s) = \frac{1}{s}$$

Therefore, solving

$$U(x,s) = c_1 e^{x\sqrt{s}} + c_2 e^{-x\sqrt{s}}$$

Since u(x,t) must be bounded as $x \to \infty$, U(x,s)=L{u(x,t)} must also be bounded as $x \to \infty$. Then we must have $c_1=0$

so that
$$U(x,s) = c_2 e^{-x\sqrt{s}}$$
Now $V(0,s)=c_2$

Therefore $U(x,s) = \dfrac{e^{-x\sqrt{s}}}{s}$

By taking inverse transform we have

$$u(x,t) = L^{-1}\{U(x,s)\} = L^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\} = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$$

Example Solve by Laplace transform method the one dimensional heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0$$

subject to

$$\frac{\partial u(0,t)}{\partial x} = -\frac{A}{k} \text{ and } u(x,0) = 0$$

Solution Taking Laplace transform of the given PDE we get

$$sV(x,s) - u(x,0) = k\frac{d^2V}{dx^2}$$

$$\Rightarrow \frac{d^2V}{dx^2} - \frac{s}{k}V = 0$$

(using initial condition)

Therefore

$$V(x,s) = c_1 e^{x\sqrt{\frac{s}{k}}} + c_2 e^{-x\sqrt{\frac{s}{k}}}$$

Since u(x,t) is bounded as $x \to \infty$ so also V(x,s), therefore we must have $c_1=0$

$$U(x,s) = c_2 e^{-x\sqrt{\frac{s}{k}}}$$

Taking Laplace transform of $\dfrac{\partial u(0,t)}{\partial x} = -\dfrac{A}{k}$ we get $\dfrac{dU(0,s)}{dx} = -\dfrac{A}{k}.\dfrac{1}{s}$

Differentiating U(x,s) with respect to x obtained above

$$\frac{dU(x,s)}{dx} = -c_2\sqrt{\frac{s}{k}}e^{-x\sqrt{\frac{s}{k}}}$$

$$\therefore \frac{dU(0,s)}{dx} = -c_2\sqrt{\frac{s}{k}}$$

$$\Rightarrow -\frac{A}{k}\frac{1}{s} = -c_2\sqrt{\frac{s}{k}}$$

$$\Rightarrow c_2 = \frac{A}{\sqrt{k}}\frac{1}{s^{\frac{3}{2}}}$$

$$\text{Therefore } U(x,s) = \frac{A}{\sqrt{k}}\frac{e^{-x\sqrt{\frac{s}{k}}}}{s^{\frac{3}{2}}}$$

Taking inverse transform we get

$$u(x,t) = L^{-1}\{U(x,s)\} = \frac{A}{\sqrt{K}}L^{-1}\left\{\frac{e^{-x\sqrt{\frac{s}{k}}}}{s^{\frac{3}{2}}}\right\}$$

$$= \frac{A}{\sqrt{K}}\left[\frac{2\sqrt{t}}{\sqrt{\pi}}e^{-\frac{x^2}{4kt}} - \frac{x}{\sqrt{k}}\operatorname{erf}c\left(\frac{x}{2\sqrt{kt}}\right)\right]$$

Example: Using the theory of Laplace transform derive the solution of the diffusion equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{K}\frac{\partial\theta}{\partial t} \quad \text{where } 0 \le x \le a,\ t > 0 \text{ and } \theta(0,t) = f(t)$$

$$0 \le x \le a,\ t > 0 \text{ and } \theta(0,t) = f(t), \theta(a,t) = 0, \theta(x,0,) = 0 .$$

Solution: Taking Lplace transform of the given PDF, we get

$$\frac{d^2\overline{\theta}}{dx^2} = \frac{1}{K}\left[S\,\overline{\theta}\,(x,S) - \theta(x,0)\right]$$

$$= \frac{S}{K}\,\overline{\theta}\,(x,S) \qquad \text{using } \theta(x,0)$$

$$\Rightarrow \frac{d^2\overline{\theta}}{dx^2} - \frac{S}{K}\,\overline{\theta} = 0 \qquad \text{where } \overline{\theta}\,(x,s) = L\{\theta(x,t)\}$$

Solution is

$$C_1 = \frac{F(s)\,e^{-a\sqrt{\frac{s}{k}}}}{e^{-a\sqrt{\frac{s}{k}}} - e^{a\sqrt{\frac{s}{k}}}},\ C_2 = \frac{F(s)\,e^{a\sqrt{\frac{s}{k}}}}{e^{a\sqrt{\frac{s}{k}}} - e^{-a\sqrt{\frac{s}{k}}}}$$

Now taking the Laplace transform of the conditions we get

$$\bar{\theta}(o,s) = L\{f(t)\} = F(s)$$

$$\bar{\theta}(a,s) = 0$$

Using these on $\bar{\theta}(x,s)$ obtained we get

$$\bar{\theta}(o,s) = C_1 + C_2$$
$$\Rightarrow F(s) = C_1 + C_2$$

and $$\bar{\theta}(a,s) = C_1 e^{a\sqrt{\frac{s}{k}}} + C_2 e^{-a\sqrt{\frac{s}{k}}}$$
$$\Rightarrow 0 = C_1 e^{a\sqrt{\frac{s}{k}}} + C_2 e^{-a\sqrt{\frac{s}{k}}}$$

Solving for C_1 and C_2 we get

$$C_1 = \frac{F(s)\, e^{-a\sqrt{\frac{s}{k}}}}{e^{-a\sqrt{\frac{s}{k}}} - e^{a\sqrt{\frac{s}{k}}}},\quad C_2 = \frac{F(s)\, e^{a\sqrt{\frac{s}{k}}}}{e^{a\sqrt{\frac{s}{k}}} - e^{-a\sqrt{\frac{s}{k}}}}$$

Therefore

$$\bar{\theta}(x,s) = \frac{F(s)}{e^{a\sqrt{\frac{s}{k}}} - e^{-a\sqrt{\frac{s}{k}}}}\left[e^{(a-x)\sqrt{\frac{s}{k}}} - e^{-(a-x)\sqrt{\frac{s}{k}}}\right] = F(s)\frac{\sinh(a-x)\sqrt{\frac{s}{k}}}{\sinh a\sqrt{\frac{s}{k}}}$$

$$\therefore \theta(x,t) = L^{-1}\{\bar{\theta}(x,s)\}$$
$$= L^{-1}\left[F(s)\frac{\sinh(a-x)\sqrt{\frac{s}{k}}}{\sinh a\sqrt{\frac{s}{k}}}\right]$$

Miscellaneous Problems

Example1.

Evaluate (i) $\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}dt$

Solution: (i) Let $f(t) = e^{-3t} - e^{-6t}$

$$L\{f(t)\} = \frac{1}{s+3} - \frac{1}{s+6} = F(s)$$

The given problem can be written as

$$\int_0^\infty e^{-st}\frac{f(t)}{t}dt \quad \text{with } s = 0$$

$$\text{i.e } L\left\{\frac{f(t)}{t}\right\} \text{ with } s = 0$$

which equals $\int_0^\infty F(s)ds$ $\left(\text{Since, } L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds\right)$

$$= \int_0^\infty \frac{ds}{s+3} - \int_0^\infty \frac{ds}{s+6} = 6$$

$$= \ln \frac{s+3}{s+6}\Bigg]_0^\infty = \ln 1 - \ln\frac{3}{6}$$

$$= \ln 2$$

Example. Find inverse Laplace transform of

$$\text{(i) } e^{-y\sqrt{s}} \quad \text{(ii) } \frac{e^{-y\sqrt{s}}}{s} \quad \text{(iii) } \frac{e^{-y\sqrt{s}}}{s^{\frac{1}{2}}} \quad \text{(ii) } \frac{e^{-y\sqrt{s}}}{s^{\frac{3}{2}}}$$

Solution: Using infinite series, we have

$$L^{-1}\left\{e^{-y\sqrt{s}}\right\} = L^{-1}\left[1 - y\sqrt{s} + \frac{y^2 s}{2!} + \frac{y^3 s^{\frac{3}{2}}}{3!} + \frac{y^4 s^2}{4!} + \cdots\right]$$

$$= L^{-1}\{1\} - yL^{-1}\left\{\sqrt{s}\right\} + \frac{y^2}{2!}L^{-1}\{s\} - \frac{y^3}{3!}L^{-1}\left\{s^{\frac{3}{2}}\right\}.$$

Using $L^{-1}\left\{s^p\right\} = 0$ for any integer p, we get

$$L^{-1}\left\{e^{-y\sqrt{s}}\right\} = \frac{yt^{-\frac{3}{2}}}{2\sqrt{\pi}} - \frac{y^3}{3!}\frac{1}{2}\frac{3}{2}\frac{t^{-\frac{5}{2}}}{\sqrt{\pi}} + \frac{y^5}{5!}\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{t^{-\frac{7}{2}}}{\sqrt{\pi}} + \cdots$$

$$= \frac{y}{2t^{\frac{3}{2}}\sqrt{\pi}}\left[1 - \frac{y^2}{2^2 t} + \frac{\left(\frac{y^2}{2^2 t}\right)^2}{2!} - \cdots\right]$$

$$= \frac{y}{2t^{\frac{3}{2}}\sqrt{\pi}} e^{-\frac{y^2}{4t}}$$

(ii) We can write immediately as $L^{-1}\left\{\frac{e^{-y\sqrt{s}}}{s}\right\} = \int_0^t \frac{ye^{-\frac{y^2}{4u}}}{2u^{\frac{3}{2}}\sqrt{\pi}}du$

$$= \frac{2}{\sqrt{\pi}}\int_{\frac{y}{2\sqrt{t}}}^\infty e^{-v^2}dv \quad (\text{setting } u = \frac{y^2}{4v^2}) = \text{erfc}\left(\frac{y}{2\sqrt{t}}\right)$$

(iii) Let $F(s) = e^{-y\sqrt{s}}$

$$\Rightarrow \frac{d}{ds}F(s) = -\frac{y}{2s^{\frac{1}{2}}}e^{-y\sqrt{s}}$$

$$\Rightarrow (-1)tf(t) = -L^{-1}\left\{\frac{y}{2s^{\frac{1}{2}}}e^{-y\sqrt{s}}\right\}$$

$$\Rightarrow tf(t) = -L^{-1}\left\{\frac{y}{2s^{\frac{1}{2}}}e^{-y\sqrt{s}}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-y\sqrt{s}}}{\sqrt{s}}\right\} = \frac{2t}{y}f(t) = \frac{2t}{y}L^{-1}\{F(s)\}$$

$$= \frac{2t}{y}\frac{y}{2t^{\frac{3}{2}}\sqrt{\pi}}e^{-\frac{y^2}{4t}}$$

$$= \frac{1}{\sqrt{\pi t}}e^{-\frac{y^2}{4t}}$$

(iii) Let $F(s) = \frac{e^{-y\sqrt{s}}}{\sqrt{s}}$

$$\Rightarrow \frac{d}{ds}F(s) = -\frac{e^{-y\sqrt{s}}}{2s^{\frac{3}{2}}} - \frac{ye^{-y\sqrt{s}}}{2s}$$

$$\Rightarrow (-1)tf(t) = -\frac{1}{2}L^{-1}\left\{\frac{e^{-y\sqrt{s}}}{s^{\frac{3}{2}}}\right\} - \frac{y}{2}L^{-1}\left\{\frac{e^{-y\sqrt{s}}}{s}\right\}$$

$$\Rightarrow 2tf(t) = L^{-1}\left\{\frac{e^{-y\sqrt{s}}}{s^{\frac{3}{2}}}\right\} + yL^{-1}\left\{\frac{e^{-y\sqrt{s}}}{s}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-y\sqrt{s}}}{s^{\frac{3}{2}}}\right\} = 2t\,f(t) - y\,L^{-1}\left\{\frac{e^{-y\sqrt{s}}}{s}\right\}$$

$$= 2tL^{-1}\{F(s)\} - yL^{-1}\left\{\frac{e^{-y\sqrt{s}}}{s}\right\}$$

$$= \frac{2t}{\sqrt{\pi t}}e^{-\frac{y^2}{4t}} - y\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right) = \frac{2\sqrt{t}}{\sqrt{\pi}}e^{-\frac{y^2}{4t}} - y\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)$$

Exercise

1. (i) Prove that $L\left\{\int_t^\infty \frac{e^{-u}}{u}du\right\} = \frac{\ln(s+1)}{s}$

(ii) $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$

2. Find the Laplace transform of

(i) $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

(ii) $f(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$

3. (i) $mX'' + bX' + KX = F(t);\quad X(0) = 0 = X'(0)$

(ii) $Y'' - tY' + Y = 1;\quad Y(0) = 1,\ Y'(0) = 2$

(iii) $tY'' + Y' + tY = 0;\quad Y(0) = 1,\ Y'(0) = 0$

(iv) $X'' + 9X = t\cos 2t;\quad X(0) = 0 = X'(0)$

(v) $X'' + 4X' = -8t;\quad X(0) = 0 = X'(0)$

(vi) $X'' + \omega^2 X' = K\sin\lambda t;\quad X(0) = 0 = X'(0)$

Unit-2

Fourier Transform Methods

Joseph Fourier, a French mathematicians, had invented a method called Fourier transform in 1801. Since then it has become a powerful tool in diverse fields of science and engineering.

Fourier Integral Representations:

Definition(Dirichlet's Conditions): A function f(x) is said to have satisfied Dirichlet's conditions in the interval(- L, L), provided f(x) in periodic, piecewise continuous and has a finite number of relative maxima and minima in (- L, L).

Let a function f(x) be periodic with period 2L, i.e. f(x + 2L) = f(x), and satisfy Dirichlets condition in the interval (- L, L). Then f(x) has a Fourier series representation for every x in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad \text{......(1)}$$

where $$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt \qquad n = 0,1,2,... \qquad \text{......(2)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt \qquad n = 1,2,... \qquad \text{......(3)}$$

Here a_n, b_n are called Fourier co-efficients. Fourier series representation, however, can be extended to some non-periodic function also, provided the integral of the modulus of such a function f(t) satisfies the condition

$$\int_{-\infty}^{\infty}|f(t)|\,dt \text{ is finite}$$

Substituting equations (*.2) and (*.3) into Fourier series (*.1), we get

$$f(x) = \frac{1}{2L}\int_{-L}^{L}f(t)dt + \sum_{n=1}^{\infty}\left[\frac{1}{L}\int_{-L}^{L}f(t)\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L}dt + \frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}dt\right]$$

Noting that cos (A - B) = cos A cos B + sin A sin B, and interchanging the order of summation and integration, we obtain

$$f(x) = \frac{1}{2L} + \int_{-L}^{L}f(t)dt + \frac{1}{L}\int_{-L}^{L}f(t)\sum_{n=1}^{\infty}\cos\frac{n\pi(t-x)}{L}dt \qquad \text{......(4)}$$

Further, if we assume that the function f(x) is absolutely integrable, and allowing L to tend to infinity, i.e.

$$\int_{-\infty}^{\infty}|f(t)|\,dt < \infty$$

We get $$\underset{L\to\infty}{Lt}\frac{1}{2L}\int_{-L}^{L}f(t)dt = 0$$

In the remaining part of the infinite sum of equation (4), if we set $\Delta s = \frac{\pi}{L}$, the equation reduces to

$$f(x) = \underset{\Delta s\to 0}{Lt}\frac{1}{\pi}\int_{-\frac{\pi}{\Delta s}}^{\frac{\pi}{\Delta s}}f(t)\sum_{n=1}^{\infty}\cos\{n\Delta s(t-x)\}\Delta s\,dt \qquad ......(5)$$

As $L \to \infty, \Delta s \to 0$ implying that Δs is a small positive number and the points $n\Delta s$ are equally spaced along the s-axis. The series under the integral can be approximated by an integral of the form (as $\Delta s \to 0$)

$$\int_{0}^{\infty}\cos\{s(t-x)\}ds$$

thus equation (5) can be written as

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\int_{0}^{\infty}\cos\{s(t-x)\}ds\,dt = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\cos\{s(t-x)\}dt\,ds$$

which is the Fourier integral representation of f(x).

Fourier Integral theorem:

Theorem 1: If f(x) satisfies Dirichlet's conditions for $-\infty < x < \infty$ and if the integral $\int_{-\infty}^{\infty}f(x)dx$ is absolutely convergent, then

$$\frac{1}{\pi}\int_{0}^{\infty}d\alpha\int_{-\infty}^{\infty}f(t)\cos\alpha(t-x)dt = \frac{1}{2}[f(x+0)+f(x-0)]$$

To establish this result, the following two Lemmas are used.

Rieman-Lebesgue Lemma: If f(x) satisfies Dirichlet's conditions in the interval (a, b), then

$$\underset{N\to\infty}{Lt}\int_{a}^{b}f(x)\sin Nx\,dx = 0$$

$$\underset{N\to\infty}{Lt}\int_{a}^{b}f(x)\cos Nx\,dx = 0$$

Riemann Localization Lemma: If f(t) satisfies Dirichlet's conditions in the interval (o, a), where a is finite, then

$$\int_0^a f(t)\,\frac{\sin Nt}{t}\,dt \to \frac{\pi}{2} f(0+) \qquad \text{as } N \to \infty$$

Proof: We may write

$$\int_0^a f(t)\,\frac{\sin Nt}{t}\,dt = f(0+)\int_0^a \frac{\sin Nt}{t}\,dt + \int_0^a \left[f(t) - f(o+)\right]\frac{\sin Nt}{t}\,dt = I_1 + I_2$$

Since the function $f'(t)$ is continuous in (o, a), from the definition of derivative,

$$\frac{f(t) - f(0+)}{t}$$

is continuous in (o, a). By the Riemann-Lebesgue Lemma(since the integrand of I_2 is bounded as $N \to \infty$) $I_2 \to 0$ as $N \to \infty$

Hence
$$\underset{N\to\infty}{Lt}\int_0^a f(t)\frac{\sin Nt}{t}\,dt = \underset{N\to\infty}{Lt}\, f(0+)\int_0^a \frac{\sin Nt}{t}\,dt = \underset{N\to\infty}{Lt}\, f(0+)\int_0^{Na} \frac{\sin u}{u}\,du$$

$$= f(0+)\int_0^{\infty} \frac{\sin u}{u}\,du = \frac{\pi}{2} f(0+).$$

Proof(Fourier Integral Theorem): Since the integral $\int_{-\infty}^{\infty} f(x)dx$ is absolutely convergent, $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite and converges for all α in the range (O, N). Also $|\cos\alpha(t-x)| \le 1$, implying that the integral

$$\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt$$

convergent and is independent of α and x. Thus, after changing the order of integration, the double integral

$$I = \int_0^N \left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\right] d\alpha \qquad \text{......(6)}$$

can be expressed as

$$I = \int_{-\infty}^{\infty}\left[\int_0^N f(t)\cos\alpha(t-x)\,d\alpha\right]dt = \int_{-\infty}^{\infty} f(t)\left[\frac{\sin N(t-x)}{t-x}\right]dt$$

Let v = t - x. then the above integral becomes

$$I = \int_{-\infty}^{\infty} f(v+x)\frac{\sin Nv}{v}\,du = \left[\int_{-\infty}^{-\delta} + \int_{-\delta}^{0} + \int_0^{\delta} + \int_{\delta}^{\infty}\right] f(v+x)\frac{\sin Nv}{v}\,du$$

$$= I_0 + I_2 + I_3 + I_4 \qquad \text{......(*)}$$

when $N \to \infty$, I_1 and I_4 both tend to zero in view of the Riemann-Lebesgue Lemma. Thus the only contribution to the integral will be from the neighbourhood of $v = 0$. Using the Riemann Localization lemma, we get

$$I_3 = \underset{N\to\infty}{Lt} \int_0^{\delta} f(v+x)\frac{\sin Nv}{v}dv = \frac{\pi}{2}f(x+)$$

and the second integral

$$I_2 = \int_{-\delta}^{0} f(v+x)\frac{\sin Nv}{v}dv = \int_0^{\delta} f(x-v)\frac{\sin Nv}{v}dv = \frac{\pi}{2}f(x-)$$

Incorporating these result into equation(6) we obtain

$$\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)dt\right]d\alpha = \frac{\pi}{2}\left[f(x+)+f(x-)\right] \quad ......(7)$$

If f(x) is a continuous function of x, then $f(x+) = f(x-) = f(x)$

and equation(7) reduces to

$$f(x) = \frac{1}{\pi}\int_0^{\infty} d\alpha\left[\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)dt\right] \quad ......(8)$$

If x is a point of discontinuity, then $f(x) = \frac{1}{2}\left[f(x+)+f(x-)\right]$

i.e. the integral (8) converges to the average value of the right and left hand limits. Thus the proof of the Fourier integral theorem is complete.

In order to bring out the analogy between Fourier series and Fourier integral theorem, we rewrite equation(8) as

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\left(\cos\alpha t\cos\alpha x + \sin\alpha t\sin\alpha x\right)dtdx$$

If we define

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\alpha t\ dt \quad ......(9)$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\alpha t\ dt \quad ......(10)$$

The above equation can also be written as

$$f(x) = \int_0^{\infty}\left[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right]d\alpha$$

Sine and cosine integral representation:

Equation (9) gives $$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x \, dx$$

If f(x) is an odd function i.e. f(- x) = - f(x), then

$$A(\alpha) = \frac{-1}{\pi}\int_{-\infty}^{\infty} f(-x)\cos\alpha x \, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x \, dx$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x \, dx = -A(\alpha)$$

implying $A(\alpha) = 0$ and

$$B(\alpha) = \frac{2}{\pi}\int_{0}^{\infty} f(+x)\sin\alpha x \, dx \qquad \text{......(11)}$$

Thus, equation (10) rerduces to

$$f(x) = \int_{0}^{\infty} B(\alpha)\sin\alpha x \, d\alpha$$

which is the Fourier sine integral representation, where $B(\alpha)$ is defined by the relation (11). Similarly, if f(x) is an even function, i.e. f(- x) = + f(x), then we obtain the Fourier cosine integral representation

$$f(x) = \int_{0}^{\infty} A(\alpha)\cos\alpha x \, d\alpha$$

where $$A(\alpha) = \frac{2}{\pi}\int_{0}^{\infty} f(x)\cos\alpha x \, dx$$

Example 1.: find Fourier-integral of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ e^{-x}, & x > o \end{cases}$$

Solution: The F-integral of f(x) is given by

$$f(x) = \int_{0}^{\infty} [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]d\alpha \quad \text{where} \quad A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} + \int_{0}^{\infty}\right] f(x)\cos\alpha x \, dx$$

$$= \frac{1}{\pi}\int_0^\infty e^{-x}\cos\alpha x\,dx = \frac{1}{\pi}\frac{1}{1+\alpha^2}$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha x dx = \frac{1}{\pi}\int_0^\infty e^{-x}\sin\alpha x\,dx = \frac{1}{\pi}\frac{\alpha}{1+\alpha^2}$$

$$\therefore \qquad f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \sin\alpha x}{1+\alpha^2}d\alpha$$

At $x = 0$,

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{1}{1+\alpha^2}d\alpha = \frac{1}{\pi}\tan^{-1}\alpha\Big|_0^\infty = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$$

Therefore, the F-integral thus represented is justified.

Example 2.: Find the F-integral of the function f(x), where

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Hence deduce that $\int_0^\infty \frac{\sin\theta}{\theta}d\theta = \frac{\pi}{2}$

Solution: The F-integral of f(x) is given by $f(x) = \int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\ ]d\alpha$

where $A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha x dx = \frac{1}{\pi}\left(\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_1^\infty\right)f(x)\cos\alpha x\,dx$

$$= \frac{1}{\pi}\int_{-1}^{1}\cos\alpha x\,dx = \frac{2}{\pi}\frac{\sin\alpha}{\alpha}$$

and $\qquad B(\alpha) = \frac{1}{\pi}\int_{-1}^{1}\sin\alpha x dx = 0 \qquad$ Thus $\quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\alpha}{\alpha}\cos\alpha x\,dx$

Now we can write

$$\int_0^\infty \frac{\sin\alpha}{\alpha}\cos\alpha x\,dx = \frac{2}{\pi}f(x) = \begin{cases} \frac{\pi}{2}\cdot 1 & |x| < 1 \\ \frac{\pi}{2}\cdot 0 & |x| > 1 \end{cases}$$

$$= \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

putting $x = 0, \alpha = \theta$ we get

$$\int_0^\infty \frac{\sin\theta}{\theta}\, d\theta = \frac{\pi}{2}$$

Example 3.: Using F-integral representation, show that

$$\int_0^\infty \frac{\sin \pi w \sin wx}{1-w^2} = \frac{\pi}{2}\sin x,\ 0 \le x \le \pi$$

$$= 0,\ x > \pi$$

Solution: By fourier sine representation for f(x), we have

$$f(x) = \int_0^\infty B(w) \sin wx\, dw$$

where $$B(w) = \frac{2}{\pi}\int_0^\infty f(x) \sin wx\, dx$$

$$= \frac{2}{\pi}\left(\int_0^\pi + \int_\pi^\infty\right) f(x) \sin wx\, dx = \frac{2}{\pi}\int_0^\pi \frac{\pi}{2}\sin x \sin wx dx = \frac{\sin w\pi}{1-w^2}$$

$$\therefore \quad f(x) = \int_0^\infty \frac{\sin w\pi}{1-w^2}\sin wxdw = \frac{\pi}{2}\sin x,\ 0 \le x \le \pi$$

$$= 0,\ x > \pi.$$

Example 4.: Show that

$$\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2}\, d\lambda = \frac{\pi}{2}e^{-x}, x > 0$$

Solution: Let $f(x) = \frac{\pi}{2}e^{-x}, x > 0$

By fourier representation for f(x), we have $f(x) = \int_0^\infty A(\lambda)\cos \lambda x\ d\lambda$

where $$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\cos \lambda x\, dx = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2}e^{-x}\cos \lambda x\, dx$$

$$= \frac{1}{1+\lambda^2}$$

$$\therefore \quad \int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda = f(x) = \frac{\pi}{2}e^{-x}, x > 0.$$

Example 5.: Show that when f(x) is an odd function

$$f(x)=\frac{2}{\pi}\int_0^\infty f(t)dt\int_0^\infty \sin ut \sin ux\, du$$

Solution: We know

$$f(x)=\int_0^\infty [A(u)\cos ux+B(u)\sin ux]du$$

where $A(u)=\frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos ut\, dt$ $B(u)=\frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin ut\, dt$

when f(x) is an odd function $A(u)=0$.

Therefore $f(x)=\int_0^\infty B(u)\sin ux\, dx$

$$=\int_0^\infty\left[\frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin utdt\right]\sin uxdu$$

$$=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty f(t)\sin utdt\right)\sin uxdu \quad \text{(since f(x) is odd)}$$

$$=\frac{2}{\pi}\int_0^\infty f(t)dt\int_0^\infty \sin ut \sin ux\, du$$

Example 6.: show that when f(x) is an even function

$$f(x)=\frac{2}{\pi}\int_0^\infty f(t)dt\int_0^\infty \cos ut \cos ux\, du$$

Solution: We know $f(x)=\int_0^\infty [A(u)\cos ux+B(u)\sin ux]du$

As in example 5. $B(u)=0$ and $A(u)=\frac{2}{\pi}\int_0^\infty f(t)\cos ut\, dt$

Therefore

$$f(x)=\frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(t)\cos utdt\right]\cos ux\, du$$

$$=\frac{2}{\pi}\int_0^\infty f(t)dt\int_0^\infty \cos ut \cos ux\, du.$$

Fourier transform Pairs:

From the Fourier Integral Theorem, we have

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\cos\alpha(t-x)\,dt\,d\alpha$$

In terms of the complex exponential function, these becomes

$$f(x)=\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\left[e^{i\alpha(t-x)}+e^{-i\alpha(t-x)}\right]dt\,d\alpha$$

$$=\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)e^{i\alpha(t-x)}dt\,d\alpha+\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)e^{-i\alpha(t-x)}dt\,d\alpha$$

Let $\alpha=-\alpha$ in the second integral, then it becomes

$$-\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)e^{i\alpha(t-x)}dt\,d\alpha=\int_{-\infty}^{0}\int_{-\infty}^{\infty}f(t)e^{i\alpha(t-x)}dt\,d\alpha$$

Hence $$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)\,e^{i\alpha(t-x)}\,dt\,d\alpha \qquad ......(12)$$

This is the exponential form of the Fourier integral theorem. Equation (12) can be rewriten as

$$f(x)=\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\,e^{i\alpha t}\,dt\right]e^{-i\alpha x}\,d\alpha \qquad ......(13)$$

Thus we define the Fourier transform pair as follows:

Definition: Let f(x) be a function defined on $(-\alpha,+\alpha)$ and is piecewise continuous, differentiable in each finite interval and is absolutely integrable on $(-\alpha,\alpha)$. From equation (13), if

$$F(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\,e^{i\alpha t}\,dt \qquad ......(14)$$

then we have, for all x

$$f(x)=\int_{-\infty}^{\infty}F(\alpha)\,e^{-i\alpha x}\,dx \qquad ......(15)$$

Here, $F(\alpha)$ defined by equation (14) is the Fourier transform of f(x) and f(x) defined by equation (15) is called the inverse Fourier transform of $F(\alpha)$ and is denoted by

$$F(\alpha)=\mathcal{F}[f(t)]$$

$$F(x)=\mathcal{F}^{-1}[F(\alpha)]$$

which constitute the Fourier transform pair.

When f(x) is an odd function then the Fourier sine transform of f(x) and its inverse sine transform is given by

$$F_s(\alpha) = \int_0^{\infty} f(t)\sin \alpha t\, dt$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\alpha)\sin \alpha x\, d\alpha = \mathcal{F}_s^{-1}[F_s(\alpha)]$$

Similarly, when f(x) is an even function, we can obtain the Fourier cosine transform and the corresponding inverse as

$$F_c(\alpha) = \int_0^{\infty} f(t)\cos \alpha t\, dt = \mathcal{F}_c[f(t)]$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\alpha)\cos \alpha x\, d\alpha = \mathcal{F}_c^{-1}[F_c(\alpha)]$$

Example : Find the Fourier transform of f(x) defined by

$$f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$$

and hence evaluate

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} d\alpha, \int_0^{\infty} \frac{\sin \alpha a}{\alpha} d\alpha$$

Solution: From the defination of the Fourier transform

$$F(\alpha) = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\, dx = \int_{-a}^{a} e^{i\alpha x}\, dx = \frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} = \frac{2}{\alpha}\left(\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i}\right)$$

Therefore

$$F(\alpha) = \begin{cases} \dfrac{2\sin \alpha a}{\alpha}, & \alpha > 0 \\ \underset{\alpha\to 0}{Lt}\ 2a\,\dfrac{\sin \alpha a}{\alpha a} = 2a, & \alpha = 0 \end{cases}$$

Now $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin \alpha a}{\alpha} e^{-i\alpha x} d\alpha = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

i.e. $$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha a\,(\cos \alpha x - i\sin \alpha x)}{\alpha} d\alpha = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Hence equating real and imaginary parts

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} d\alpha = \begin{cases} \pi, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Also by setting x = 0 in the above equation, we obtain $\int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha} d\alpha = \pi$

Since the integrand is even, we can have $\int_{0}^{\infty} \frac{\sin \alpha a}{\alpha} d\alpha = \frac{\pi}{2}$

In particular putting a = 1, we can have $\int_{0}^{\infty} \frac{\sin \alpha a}{\alpha} d\alpha = \frac{\pi}{2}$

Example : Find the Fourier cosine and sine transforms of e^{-bx} and evaluate tha integrals

(i) $\int_{0}^{\infty} \frac{\cos \alpha x}{\alpha^2 + b^2} d\alpha$ (ii) $\int_{0}^{\infty} \frac{\alpha \sin \alpha x}{\alpha^2 + b^2} d\alpha$

Solution: Given $f(x) = e^{-bx}$ and following the definitions of Fourier cosine and sine transforms, viz.

$$F_c(\alpha) = \mathcal{F}_c[f(x)] = \int_0^{\infty} f(x) \cos \alpha x \, dx$$

$$F_s(\alpha) = \mathcal{F}_s[f(x)] = \int_0^{\infty} f(x) \sin \alpha x \, dx$$

we obtain $\mathcal{F}_c\left[e^{-bx}\right] = \int_0^{\infty} e^{-bx} \cos \alpha x \, dx$

$$\mathcal{F}_s\left[e^{-bx}\right] = \int_0^{\infty} e^{-bx} \sin \alpha x \, dx$$

Let $I_1 = \int_0^{\infty} e^{-bx} \cos \alpha x \, dx$, $I_2 = \int_0^{\infty} e^{-bx} \sin \alpha x \, dx$. Integrating I_1 by parts, we have

$$I_1 = \left(-\frac{1}{b} e^{-bx} \cos \alpha x \right)_0^{\infty} - \frac{\alpha}{b} \int_0^{\infty} e^{-bx} \sin \alpha x \, dx = \frac{1}{b} - \frac{\alpha}{b} I_2$$

Similarly integral I_2 by parts, we have $I_2 = \frac{\alpha}{b} I_1$

solving for I_1 and I_2 from above we get $I_1 = \frac{b}{\alpha^2 + b^2}$, $I_2 = \frac{\alpha}{\alpha^2 + b^2}$

Hence $F_c(\alpha) = \frac{b}{\alpha^2 + b^2}$, $F_s(\alpha) = \frac{\alpha}{\alpha^2 + b^2}$

Then $$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(\alpha)\cos\alpha x \, d\alpha$$

i.e. $$e^{-bx} = \frac{2}{\pi}\int_0^{\infty} \frac{b}{\alpha^2 + b^2}\cos\alpha x \, d\alpha \quad \text{or} \quad \int_0^{\infty} \frac{\cos\alpha x}{\alpha^2 + b^2} d\alpha = \frac{\pi}{2b} e^{-bx}$$

Similarly, it can be shown that

$$\int_0^{\infty} \frac{\alpha \sin\alpha x}{\alpha^2 + b^2} d\alpha = \frac{\pi}{2} e^{-bx}$$

Example: Find the Fourier transform of

$$f(x) = 1 - x^2, |x| < 1$$
$$= 0, |x| > 1$$

and hence evaluate $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2} \, dx$

Solution: $$F(\alpha) = \mathcal{F}[f(x)] = \int_{-1}^{1} \left(1 - x^2\right) e^{i\alpha x} dx = \frac{4}{\alpha^3}(\sin\alpha - \alpha\cos\alpha)$$

Then $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x} \, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{\alpha^3}(\sin\alpha - \alpha\cos\alpha) e^{-i\alpha x} \, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{\alpha^3}(\sin\alpha - \alpha\cos\alpha)(\cos\alpha x - i\sin\alpha x) \, d\alpha$$

$$\therefore \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{\alpha^3}(\sin\alpha - \alpha\cos\alpha)(\cos\alpha x - i\sin\alpha x) \, d\alpha = f(x)$$

$$= \begin{cases} 1 - x^2 & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Equating real parts we get

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha)\cos\alpha x}{\alpha^3} d\alpha = \begin{cases} 1 - x^2 & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha)\cos\alpha x}{\alpha^3} d\alpha = \begin{cases} \frac{\pi}{2}\left(1-x^2\right) & |x|<1 \\ 0, & |x|>1 \end{cases}$$

Putting $x = \frac{1}{2}$

$$\int_{-\infty}^{\infty} \frac{(\sin\alpha - \alpha\cos\alpha)}{\alpha^3} \cos\frac{\alpha}{2} d\alpha = \frac{\pi}{2}\left(1-\frac{1}{4}\right) = \frac{3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\alpha\cos\alpha - \sin\alpha}{\alpha^3} \cos\frac{\alpha}{2} d\alpha = -\frac{3\pi}{8}$$

Since the integral is even, hence

$$2\int_{0}^{\infty} \frac{\alpha\cos\alpha - \sin\alpha}{\alpha^3} \cos\frac{\alpha}{2} d\alpha = -\frac{3\pi}{8}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\alpha\cos\alpha - \sin\alpha}{\alpha^3} \cos\frac{\alpha}{2} d\alpha = -\frac{3\pi}{16}$$

Replacing α by x we get the required result.

Properties of Fourier transform:

Theorem 2.(Linearity property): If $F(\alpha)$ and $G(\alpha)$ one the Fourier transform of f(x) and g(x) respectively, then

$$\mathcal{F}[c_1 f(x) + c_2 g(x)] = c_1 F(\alpha) + c_2 G(\alpha)$$

$$\mathcal{F}^{-1}[c_1 F_1(\alpha) + c_2 G(\alpha)] = c_1 f(x) + c_2 g(x)$$

where c_1, c_2 are contants (proof is left to reader)

Theorem 3.(Change of Scale): If $F(\alpha)$ is the Fourier transform of f(x), then the Fourier transform of f(ax) is $\frac{1}{a}F\left(\frac{\alpha}{a}\right)$. (proof is left to reader)

Theorem 4.(Shifting Property): If $F(\alpha)$ is the Fourier transform of f(x), then Fourier transform of $f(x - a)$ is $e^{+ia\alpha}F(\alpha)$. (proof is left to reader)

Theorem 5.(Differentiation): If f(x) and its first (r - 1) deriva tives are continuous, and if its r^{th} derivative is piecewise continuous, then

$$\mathcal{F}[f^{(r)}(x)] = (-i\alpha)^r \mathcal{F}[f(x)] \qquad r = 0, 1, 2, \ldots\ldots$$

provided f and its derivatives are ab solutely integrable. In addition, we assume that f(x) and its first (r - 1) derivatives vanishes as $x \to \pm\infty$.

Proof: From the definition, we have the Fourier transform of $\frac{d^r f}{dx^r}$ as

$$\mathcal{F} = \left[\frac{d^r f}{dx^r}\right] = \int_{-\infty}^{\infty} \frac{d^r f}{dx^r} e^{i\alpha x}\, dx = F^{(r)}(\alpha) \qquad \text{(say)}$$

Integrating by parts, we get

$$\int_{0}^{\infty} \frac{d^r f}{dx^r} e^{i\alpha x}\, dx = \left(\frac{d^{r-1} f}{dx^{r-1}} e^{i\alpha x}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (i\alpha) e^{i\alpha x} \frac{d^{r-1} f}{dx^{r-1}} dx$$

If we assume that $\frac{d^{r-1} f}{dx^{r-1}}$ tends to zero as $x \to \pm\infty$, we may write the above result in the form

$$F^{(r)}(\alpha) = (-i\alpha)^{r} F^{(r-1)}(\alpha) = (-i\alpha)^2 F^{(r-2)}(\alpha) = \ldots\ldots = (-i\alpha)^r F(\alpha)$$

Hence $\quad F^{(r)}(\alpha) = (-i\alpha)^r F(\alpha)$

and therefore $\quad \mathcal{F}[f^{(r)}(x) = (-i\alpha)^r F(\alpha)$

Convolution Theorem(Faltung Theorem):

If $F(\alpha)$ and $G(\alpha)$ one the Fourier transforms of the functions $f(x)$ and $g(x)$, then the product $F(\alpha)$ $G(\alpha)$ is the Fourier transform of the convolution product $f * g$.

Proof: The product $\quad f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$

is called the convolution or Faltung of the function f and g over the interval $(-\infty, \infty)$. Than the Fourier transform of this convolution integral yields.

$$\mathcal{F}(f * g) = \int_{-\infty}^{\infty} e^{i\alpha x} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\alpha x} f(u)\, g(x-u)\, du\, dx$$

Since f and g are absolutely integrable, the order of integration can be interchanged and therefore

$$\mathcal{F}(f * g) = \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(x-u) e^{i\alpha(x-u)} e^{i\alpha u}\, dx\right] du$$

Let $x - u = \gamma$. Then $dx = d\gamma$, therefore

$$\mathcal{F}(f * g) = \int_{-\infty}^{\infty} f(u) \left[e^{i\alpha u} \int_{-\infty}^{\infty} g(\gamma) e^{i\alpha\gamma} d\gamma\right] du = \int_{-\infty}^{\infty} e^{i\alpha u} f(u)\, du \int_{-\infty}^{\infty} e^{i\alpha\gamma} g(\gamma) d\gamma = F(\alpha)\, G(\alpha)$$

Hence the theorem is proved

It can be verified that $f * g = g * f$.

Parseval's Relation:

If $F(\alpha)$ and $G(\alpha)$ are complex Fourier transform of $f(x)$ and $g(x)$ respectively, then

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\overline{G(\alpha)}\,d\alpha = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\,d\alpha = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

where bar represents the complex conjugate.

Proof: Using the inversion formula for Fourier transform, we get

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)e^{-i\alpha x}\,d\alpha$$

Taking conjugate complex of both sides wee get

$$\overline{g(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(\alpha)}e^{i\alpha x}\,d\alpha$$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} f(x)\,dx\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G(\alpha)}e^{i\alpha x}\,d\alpha\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{G(\alpha)}\,e^{i\alpha x}\,dx\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\overline{G(\alpha)}\,e^{i\alpha x}\,d\alpha\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G(\alpha)}\,d\alpha\left[\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G(\alpha)}\,d\alpha\,\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\overline{G(\alpha)}\,d\alpha$$

Putting $g(x) = f(x)$ i.e. $G(\alpha) = F(\alpha)$, we get

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\overline{F(\alpha)}\,d\alpha$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\,d\alpha$$

Example: Show that $\int_{0}^{\infty}\frac{\sin^2 u}{u^2}\,du = \frac{\pi}{2}$

Solution: Let $f(x)=\begin{cases}1 & |x|<a\\ 0 & |x|>a\end{cases}$

The Fourier transform of f(x) is

$$\mathcal{F}\{f(x)\}=\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx=\int_{-a}^{a} e^{i\alpha x} dx \quad \text{i.e.} \quad F(\alpha)=\frac{2\sin \alpha a}{\alpha}$$

Using Parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx=\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha$$

$$\Rightarrow \int_{-a}^{a} dx=\frac{1}{2\pi} 4 \int_{-\infty}^{\infty} \frac{\sin^2 \alpha a}{\alpha^2} d\alpha \Rightarrow 2a=\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 \alpha a}{\alpha^2} d\alpha$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 \alpha a}{\alpha^2} d\alpha=\pi a \Rightarrow \int_{0}^{\infty} \frac{\sin^2 \alpha a}{\alpha^2} d\alpha=\frac{\pi a}{2}$$

Putting $\alpha a=u$ we get $d\alpha=\frac{du}{a}$. Hence the integral becomes

$$\int_{0}^{\infty} a^2 \frac{\sin^2 u}{u^2}\frac{du}{a}=\frac{\pi a}{2} \quad \Rightarrow \int_{0}^{\infty} \frac{\sin^2 u}{u^2} du=\frac{\pi}{2}$$

Example: Find the Fourier transform of

(i) $\frac{\partial^n u}{\partial x^n}$ of the function u(x, t) assuming that u and its first (n - 1) derivatives w.r.t. x vanish as $x \to \pm\infty$.

(ii) $\frac{\partial u}{\partial t}$

Solution: (i) We shall adopt the following notation:

The Fourier transform of u(x, t) w.r.t. the variable x ix defined as

$$\mathcal{F}[u(x,t)]=\int_{-\infty}^{\infty} e^{i\alpha x} u(x,t)\, dx=U(\alpha,t)$$

Then the Fourier transform of $\frac{\partial u}{\partial x}$ is $\mathcal{F}\left[\frac{\partial u(x,t)}{\partial x}\right]=\int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\alpha x}\, dx$

Integration by parts yields

$$\left[u(x,t)e^{i\alpha x}\right]_{-\infty}^{\infty}-i\alpha\int_{-\infty}^{\infty} u(x,t)e^{i\alpha x}\, dx$$

If we assume $\underset{x \to \pm\infty}{Lt} u(x,t) = 0$

Then we find that $\mathcal{F}\left[\frac{\partial u(x,t)}{\partial x0}\right] = -i\alpha\, \mathcal{F}[u(x,t)] = -i\alpha \cup(\alpha,t)$

Similarly, the Fourier transform of $\frac{\partial^2 u}{\partial x^2}$ is $(i\alpha)^2 \cup(\alpha,t)$, assuming that both u and $\frac{\partial u}{\partial x}$ tend to zero as $x \to \pm\infty$.

Thus in general the Fourier transform of the n th derivative of u(x, t) is given by

$$\mathcal{F}\left[\frac{\partial^n u(x,t)}{\partial x^n}\right] = (-1)^n (i\alpha)^n \cup(\alpha,t)$$

(ii) Now

$$\mathcal{F}\left[\frac{\partial u(x,t)}{\partial t}\right] = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\alpha x}\, d\alpha = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u\, e^{i\alpha x}\, d\alpha = \frac{\partial U(\alpha,t)}{\partial t}.$$

Example(Solution of diffusion equation): Solve the heat conduction equation given by

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \; -\infty < x < \infty, t > 0$$

subject to u(x, t) and $u_x(x, t)$ both $\to 0$ as $|x| \to \infty$ and $u(x, 0) = f(x)$, $-\infty < x < \infty$.

Solution: Taking the Fourier transform of the PDE, we get

$$k(i\alpha)^2 \cup(\alpha, t) = U_t(\alpha, t)$$

or $\quad U_t(\alpha, t) + k\alpha^2 \cup(\alpha, t) = 0.$

Solution is $\quad U(\alpha, t) = Ae^{-k\alpha^2 t}$

Now the Fourier transform of the initial condition is given by

$$\mathcal{F}[u(x, 0)] = \mathcal{F}[f(x)]$$

i.e. $\quad U(\alpha, 0) = F(\alpha), -\infty < \alpha < \infty.$

Using this $A = F(\alpha)$, hence $\quad U(\alpha,t) = F(\alpha)e^{-k\alpha^2 t}$

Taking inverse transform we get $\quad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{-k\alpha^2 t - i\alpha x}\, d\alpha$$

The above equation suggests the use of convolution. If the Fourier transform of g(x) is $e^{-k\alpha^2 t}$,

Then g(x) will be given by

$$g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k\alpha^2 t}\, e^{-i\alpha x}\, d\alpha$$

But, we know, iF $a>0$, b is real or complex, that

$$\int_{-\infty}^{\infty} e^{-ax^2-2bx}\, dx=\frac{\sqrt{\pi}}{\sqrt{a}}e^{\frac{b^2}{a}}$$

Here $a = kt$, $2b = i\alpha$, therefore $g(x)=\frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{kt}}e^{-\frac{x^2}{4kt}}=\frac{1}{2\sqrt{\pi kt}}e^{-\frac{x^2}{4kt}}$

Therefore using the convolution theorem, we have

$$u(x,t)=\int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)d\alpha=\int_{-\infty}^{\infty} f(\alpha)\frac{1}{2\sqrt{\pi kt}}e^{\frac{-(x-\alpha)^2}{4kt}}\, d\alpha$$

$$=\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(\alpha)e^{\frac{-(x-\alpha)^2}{4kt}}\, d\alpha.$$

Example:(Solution of wave equation):

Compute the displecement u(x, t) of an infinite string using the method of Fourier transform given that the string in initially at rest that the initial displacement is f(x), $-\infty<x<\infty$.

Solution: Displacement of an infinite string is governed by the PDE

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2},\ -\infty<x<\infty$$

and initial conditions $u(x, 0) = f(x), -\infty<x<\infty$, $u_t(x, 0) = 0$.

Taking the Fourier transform, we have

$$\frac{d^2U(\alpha,t)}{dt^2}=c^2(i,\alpha)^2 \cup(\alpha,t) \quad \text{or} \quad \frac{d^2u}{dt^2}+c^2\alpha^2U=0$$

whose solution is $U(\alpha, t) = A\cos(c\alpha t) + B\sin(c\alpha t)$

The Fourier transform of the initial conditions gives $\frac{dU}{dt}(\alpha,t)=0, U=F(\alpha)$

i.e. $(Ac\alpha\sin(c\alpha t) + Bc\alpha\cos(c\alpha t))t = 0$ implying $Bc\alpha = 0$ or $B = 0$. Also $U(\alpha, 0) = F(\alpha)$

$$\Rightarrow A = F(\alpha)$$

Thus $U(\alpha, t) = F(\alpha) \times \cos(c\alpha t)$

Taking its inverse Fourier transform, we obtain

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\cos(c\alpha t)e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\frac{e^{+ic\alpha t} + e^{-ic\alpha t}}{2}e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\frac{e^{-i\alpha(+x+ct)} + e^{-i\alpha(x-ct)}}{2}\, d\alpha$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha(x+ct)}\, d\alpha + \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha(x-ct)}\, d\alpha\right]$$

Using the result $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)e^{-i\alpha x}\, d\alpha$ we get $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$

Example:(Solution of Laplace equation): Solve the following boundary value problem in the half plane $y > 0$, discribed by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0,\ -\infty < x < \infty,\ y > 0 \quad \text{with} \quad u(x,0) = f(x),\ -\infty < x < \infty,$$

u is bounded as $y \to \infty$, u and $\frac{\partial u}{\partial x}$ both vanish as $|x| \to \infty$

Solution: Since x has an infinite range of values, we take the Fourier exponential transform of the PDE in the variable x to get

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] + \mathcal{F}\left[\frac{\partial^2 u}{\partial y^2}\right] = 0$$

Since u and $\frac{\partial u}{\partial x}$ both vanish as $|x| \to \infty$, we have

$$-\alpha^2 U(\alpha, y) + \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2} e^{+i\alpha x}\, dx = 0$$

or $$-\alpha^2 U(\alpha, y) + \frac{\partial^2}{\partial y^2}\left[\int_{-\infty}^{\infty} u(x,y)e^{+i\alpha x}\, dx\right] = 0$$

or $$\frac{\partial^2 U(\alpha, y)}{\partial y^2} - \alpha^2 U(\alpha, y) = 0.$$

whose solution is given by

$$\cup(\alpha, y) = A(\alpha)e^{\alpha y} + B(\alpha)e^{-\alpha y}$$

Since u must be bounded as $y \to \infty$, $\cup(\alpha, y)$ should be bounded as $y \to \infty$, implying $A(\alpha) = 0$ for $\alpha > 0$ but if $\alpha < 0$, $B(\alpha) = 0$, thus for any α

$$\cup(\alpha, y) = \text{constant}\left(e^{-|\alpha|y}\right)$$

Now the Fourier transform of the initial con dition yields

$$\cup(\alpha, 0) = F(\alpha)$$

Thus using this we get $F(\alpha)$ = constant.

Hence $$\cup(\alpha, y) = F(\alpha)e^{-|\alpha|y} = \int_{-\infty}^{\infty} f(x)e^{-|\alpha|y}e^{i\alpha x}dx$$

Taking Fourier inverse transform, we obtain, after replacing the dummy variable x by ξ, the equation

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi)e^{-|\alpha|y}e^{i\alpha\xi}\,d\xi\right]e^{i\alpha x}d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi\int_{-\infty}^{\infty} e^{\alpha(i\xi - x) - |\alpha|y}\,d\alpha$$

But $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\alpha i(\xi - x) - |\alpha|y}\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{\alpha\{y + i(\xi - x)\}}d\alpha + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\alpha\{y - i(\xi - x)\}}d\alpha$$

$$= \frac{1}{2\pi}\left[\frac{e^{\alpha\{y + i(\xi - x)\}}}{y + i(\xi - x)}\right]_{-\infty}^{0} - \frac{1}{2\pi}\left[\frac{e^{-\alpha\{y - i(\xi - x)\}}}{y - i(\xi - x)}\right]_{0}^{\infty}$$

$$= \frac{1}{2\pi}\left[\frac{1}{y + i(\xi - x)} + \frac{1}{y - i(\xi - x)}\right] = \frac{1}{\pi}\frac{y}{(\xi - x)^2 + y^2}$$

Therfore

$$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{(\xi - x)^2 + y^2}$$

This solution is a well known Poisson integral formula and is valid for $y > 0$, when f(x) is bounded and piecewise continuous for all real x.

Transform of Dirac delta function:

The Dirac delta function has been defined in chapter. We may recall the shifting property of the delta function, i.e.,

$$\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a)$$

and then obtain its Fourier transform as $\mathcal{F}[\delta(t-a)] = \int_{-\infty}^{\infty} \delta(t-a)e^{i\alpha t}dt = e^{i\alpha a}$

when a = 0, we obtain the formal result $\mathcal{F}[\delta(t)] = 1$

That is, the Fourier transform of the Dirac delta function $\delta(t)$ is constant and equal to 1. It then follows that

$$\mathcal{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha t}(1)\,d\alpha = \delta(t)$$

Relationship between Fourier transform and Leplace Transform:

Let us define a function

$$f(t) = e^{-xt}\phi(t),\ t > 0$$
$$= 0,\ t < 0$$

Now $\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{+iyt}dt = \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) f(t)e^{+iyt}dt$

$$= \int_{0}^{\infty} e^{-xt}\varphi(t)e^{iyt}dt = \int_{0}^{\infty} e^{-(x-iy)t}\varphi(t)dt$$

$$= \int_{0}^{\infty} e^{-st}\varphi(t)dt, \qquad \text{(setting } x - iy = s\text{)}$$

$$= L\{\varphi(t)\}.$$

Exercise

1. Show that the Fourier cosine representation of the function f(x) defined by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0 & x > 2 \end{cases} \text{ is}$$

$$\frac{2}{\pi}\int_0^\infty \frac{2\cos\lambda-\cos 2\lambda-1}{\lambda^2}\cos\lambda x\ d\lambda$$

2. Using Fourier integral representation, show that

$$\int_0^\infty \frac{\cos\left(\pi\frac{w}{2}\right)\cos xw}{1-w^2}dw = \begin{cases}\frac{\pi}{2}\cos x & |x|<\frac{\pi}{2}\\ 0 & |x|>\frac{\pi}{2}\end{cases}$$

3. Solve the integral equation

$$\int_0^\infty F(x)\cos\lambda x\ d\lambda = 1-\lambda, 0\le\lambda\le 1$$

$$= 0,\ \lambda>1.$$

4. Find the Fourier transfom of

$$f(x)=e^{-\frac{x^2}{2}}.$$

■■■

Unit-3

Linear Integral equation

Definition: An integral equation is an equation in which an unknown function appears under one or more integral signs. Naturally, in such an equation there can occur others terms as well. For example, for $a \le s \le b$, $a \le t \le b$, the equations

$$f(s) = \int_a^b K(s,t)g(t)dt$$

$$g(s) = f(s) + \int_a^b K(s,t)g(t)dt$$

$$g(s) = \int_a^b K(s,t)[g(t)]^2 dt$$

where the function g(s) is the unknown function and all the other functions are known, are integral equations. A differential equation can be replaced by an integral equation that incorporates its boundary conditions.

An integral equation is called linear if only linear operations are performed in it upon the unknown function. The first two equations above are linear, while the third is non-linear. In this chapter we deal only linear integral equations.

The most general type of linear integral equation is of the form

$$h(s)g(s) = f(s) + \lambda \int_a K(s,t)g(t)dt = 0 \qquad ....(1)$$

where the upper limit may be either variable or fixed. The function f,h and K are known functions whereas g is to be determined; λ is a nonzero, real or complex parameter. The function K(s,t) is called the kernel. The following special cases of equation (1) are of main interest.

(i) **Fredholm integral equations:** In all Fredholm integral equations, the upper limit of integration b, say is fixed.

(a) In the Fredholm integral equation of the first kind h(s) = 0, thus

$$f(s) + \lambda \int_a^b k(s,t)g(t)dt = 0$$

(b) In the Fredholm integral equation of the second kind, h(s)=1;

$$g(s) = f(s) + \lambda \int_a^b K(s,t)g(t)dt$$

(c) The homogeneous Fredholm integral equation of the second kind is a special case of (b) above. In this case f(s) = 0:

$$g(s) = \lambda \int_a^b K(s,t)g(t)dt$$

(ii) Volterra equations: Volterra equations of the first, homogeneous and second kind are defined precisely as above except that b = s is the variable upper limit of integration.

Equation (1) itself is called the Carleman type integral equation. It is also called the integral equation of the third kind.

(iii) Singular integral equations: When one or both limits of integration becomes infinite or when the kernel approaches infinity at one or more points within the range of integration, the integral equation is called singular. For example, the integral equation

$$g(s) = f(s) + \lambda \int_{-\infty}^{\infty} e^{-|s-t|} g(t)dt$$

and $$f(s) = \int_0^s \frac{1}{(s-t)^a} g(t)\,dt, \quad 0 < a < 1$$

are singular integral equations.

In this chapter, we shall not deal with singular equations.

Special kinds of Kernels:

(i) Separable or degenerate kernel : A kernel K(s,t) is called separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of s only and a function of t only, that is

$$K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$$

The functions $a_i(s)$ can be assumed to be linearly independent, otherwise the number of terms in the above relation can be reduced.

(ii) Symmetric kernel : A complex valued function K(s,t) is called symmetric (or Hermitian) if K(s,t) = $K^*(t,s)$, where the asterisk denotes complex conjugate. For a real kernel, K(s,t)=K(t,s).

Integral equation with separable kernels:

Reduction to a system of algebraic equations:

We have defined already that a degenerate or a separable kernel as

$$K(s,t) = \sum_{i=1}^{n} a_i(s) b_i(t)$$

where the functions $a_1(s), \ldots a_n(s)$ and the functions $b_1(t), \ldots b_n(t)$ are linearly independent. With such a kernel, the Fredholm integral equation of the second kind

$$g(s) = f(s) + \lambda \int K(s,t) g(t)dt \quad \text{becomes}$$

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} a_i(s) \int b_i(t) g(t)\,dt \qquad \text{....(2)}$$

It emerges that the technique of solving this equation is essentially dependent on the choice of the complex parameter λ and on the definition of

$$C_i = \int b_i(t) g(t) dt \qquad \text{....(3)}$$

The quantities C_i are constants, although unknown. Substituting equation (3) in (2) gives

$$g(s) = f(s) + \lambda \sum_{i=1}^{n} C_i a_i(s) \qquad \text{....(4)}$$

and the problem reduces to finding the quantities C_i.

Now we multiply both sides of (4) by $b_1(s), b_2(s), \ldots b_n(s)$ and integrating to get

$$C_i = \int b_i(t)f(t)dt + \lambda\sum_{K=1}^{n} C_K \int a_K(t)b_i(t)dt \qquad \text{....(5)}$$

Using the simplified notation

$$\int b_i(t)f(t)dt = f_i, \quad \int b_i(t)a_K(t)dt = a_{iK}$$

where f_i and a_{iK} are known constants.

Equation (5) becomes

$$C_i - \lambda\sum_{K=1}^{n} a_{iK}c_K = f_i, i = 1, \ldots n \qquad \text{....(6)}$$

that is, a system of n algebraic quantities for the unknown C_i. The determinant $D(\lambda)$ of this system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1-\lambda a_{nn} \end{vmatrix}$$

which is a polynomial in λ of degree at most n. Moreover it is not identically zero, since, when $\lambda = 0$, it reduces to unity.

For all values of λ for which $D(\lambda) \neq 0$, the algebraic system (6) and thereby the integral equation (1), has a unique solution. These values of λ are called regular. On the other hand for all values of λ for which $D(\lambda)$ becomes equal to zero, the algebraic system (6) and with it the integral equation (1), either insolvable or has a infinite number of solutions. Setting $\lambda = \frac{1}{\mu}$ in equation (6), we have the eigen value problem of matrix theory. The eigen values are given by the polynomial $D(\lambda)=0$. They are also the eigen values of our integral equation.

Example : Solve

$$g(s) = s + \lambda\int_0^1 (st^2 + s^2t)g(t)\,dt$$

Solution : The Kernel $K(s,t)=st^2+s^2t$ is separable and we can set

$$C_1 = \int_0^1 t^2 g(t)\,dt, \qquad C_2 = \int_0^1 t g(t)\,dt$$

then the given equation becomes

$$g(s) = s + \lambda C_1 s + \lambda C_2 s^2 \qquad \text{....(1)}$$

Multiplying this equation by t^2, t and integrating over (0, 1) we get

$$C_1 = \frac{1}{4} + \frac{1}{4}\lambda C_1 + \frac{1}{5}\lambda C_2$$

$$C_2 = \frac{1}{3} + \frac{1}{3}\lambda C_1 + \frac{1}{4}\lambda C_2.$$

The solution of these equations is readily obtained as

$$C_1 = \frac{60+\lambda}{240-120\lambda-\lambda^2}$$

$$C_2 = \frac{80}{240-120\lambda-\lambda^2}$$

Thus required solution for (1) is then

$$g(s) = \frac{(240-60\lambda)s + 80\lambda s^2}{240-120\lambda-\lambda^2}$$

Example 2: Solve

$$g(s) = f(s) + \lambda \int_0^1 (s+t) g(t) dt$$

and find the eigenvalues.

Solution: Here

$$a_1(s) = s \qquad a_2(s) = 1$$

$$b_1(t) = 1 \qquad b_2(t) = t$$

$$a_{11} = \int_0^1 t\,dt = \frac{1}{2}, \quad a_{12} = \int_0^1 dt = 1$$

$$a_{21} = \int_0^1 t^2\,dt = \frac{1}{3} \quad a_{22} = \int_0^1 t\,dt = \frac{1}{2}$$

$$f_1 = \int_0^1 f(t)\,dt, \quad f_2 = \int_0^1 t f(t)\,dt$$

Substituting these values in equation

$$C_i - \lambda \sum_{K=1}^{n} a_{iK} C_K = f_i, \quad i = 1,2$$

we have

$$\left(1 - \frac{1}{2}\lambda\right)C_1 - \lambda C_2 = f_1$$

$$-\frac{1}{3}\lambda C_1 + \left(1 - \frac{1}{2}\lambda\right)C_2 = f_2$$

The determinant $D(\lambda)=0$ gives

$$\lambda^2 + 12\lambda - 12 = 0.$$

Thus the eigenvalues are

$$\lambda_1 = \left(-6+4\sqrt{3}\right) \qquad \lambda_2 = \left(-6-4\sqrt{3}\right)$$

For these two values of λ, the homogenous equation has a nontrivial solution, whereas the given integral equation is, in general, not solvable. When λ differes from these values, the solution of the preceding algebraic system is

$$C_1 = \frac{-12f_1 + \lambda(6f_1 - 12f_2)}{\lambda^2 + 12\lambda - 12}$$

$$C_2 = \frac{-12f_2 + \lambda(4f_1 - 6f_2)}{\lambda^2 + 12\lambda - 12}$$

Thus the relation

$$g(s) = f(s) + \lambda \sum_{i=1}^{2} C_i a_i(s)$$

becomes

$$g(s) = f(s) + \lambda \int_0^1 \frac{6(\lambda-2)(s+t) - 12\lambda st - 4\lambda}{\lambda^2 + 12\lambda - 12} f(t)\,dt$$

The function $\Gamma(s,t;\lambda)$

$$\Gamma(s,t;\lambda) = \frac{6(\lambda-2)(s+t) - 12\lambda st - 4\lambda}{\lambda^2 + 12\lambda - 12}$$

is called the resolvent kernel. We have therefore succeeded in inverting the integral equation because the right hand side of the preceding formula is a known quantity.

●●●

Method of successive approximation:

Ordinary first order differential equation can be solved by the well known Picard method of successive approximation. An iterative scheme based on the same principle is also available for linear integral equations of the second kind;

$$g(s) = f(s) + \lambda \int K(s,t)\, g(t)\, dt \qquad \text{....(7)}$$

We assume the function f(s) and K(s,t) are square integrable i.e.

$$\int |f(s)|^2 ds < \infty$$

$$\int |K(s,t)|^2 dt < \infty,$$

$$\int |K(s,t)|^2 ds < \infty$$

As a zero order approximation to the desired function g(s), the solution $g_0(s)$,

$$g_0(s) = f(s)$$

is taken. This is substituted into the right hand side of equation (7) to give the first order approximation

$$g_1(s) = f(s) + \lambda \int K(s,t)\, g_0(t)\, dt$$

This function when substituted into equation (7) yields the second approximation. This process is then repeated; the (n+1)th approximation is obtained by substituting the nth approximation in the right hand side of (7). These results the recurrence relation

$$g_{n+1}(s) = f(s) + \lambda \int K(s,t)\, g_n(t)\, dt$$

If $g_n(s)$ tends uniformly to a limit as $n \to \infty$, then this limit is the required solution. To study such a limit, let us examine the iterative procedure in detail. The first and second order approximations are

$$g_1(s) = f(s) + \lambda \int K(s,t)\, f(t)\, dt$$

and

$$g_2(s) = f(s) + \lambda \int K(s,t) f(t)\, dt + \lambda^2 \int K(s,t) \left[\int K(t,x) f(x)\, dx \right] dt$$

This formula can be simplified by setting

$$K_2(s,t) = \int K(s,x)\, K(x,t)\, dx$$

and by changing the order of integration.

The result is

$$g_2(s) = f(s) + \lambda \int K(s,t) f(t)\, dt + \lambda^2 \int K_2(s,t) f(t)\, dt$$

Similarly

$$g_3(s) = f(s) + \lambda \int K(s,t) f(t)\, dt + \lambda^2 \int K_2(s,t) f(t)\, dt + \lambda^3 \int K_3(s,t) f(t)\, dt$$

where $K_3(s,t) = \int K(s,x) K_2(x,t) dx$

By continuing this process, and denoting

$$K_m(s,t) = \int K(s,x) K_{m-1}(x,t) dx$$

we get the nth approximate solution of the integral equation as

$$g_n(s) = f(s) + \sum_{m=1}^{n} \lambda^m \int K_m(s,t) f(t)\,dt$$

We call the expression $K_m(s,t)$ the mth iterate, where $K_1(s,t)=K(s,t)$. Passing to the limit as $n \to \infty$, we obtain the so-called Neumann series

$$g(s) = \underset{n\to\infty}{Lt}\, g_n(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int K_m(s,t) f(t)\,dt \qquad \text{....(8)}$$

Now, let us evaluate the resolvent kernel in terms of the iterate kernels $K_m(s,t)$. Indeed by changing the order of integration and summation in the Neumann series, we obtain

$$g(s) = f(s) + \lambda \int \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t) \right] f(t)\,dt$$

$$= f(s) + \lambda \int \Gamma(s,t,\lambda) f(t)\,dt \qquad \text{....(9)}$$

where $\quad \Gamma(s,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(s,t).$

Example : Solve $\quad g(s) = f(s) + \lambda \int_0^1 e^{s-t} g(t)\,dt.$

Solution : Here

$$K_1(s,t) = e^{s-t}$$

$$K_2(s,t) = \int_0^1 e^{s-x} e^{x-t}\,dx = e^{s-t}.$$

Proceeding in this way, we find that all the iterated kernels coincide with K(s,t). Thus the resolvent kernel is

$$\Gamma(s,t;\lambda) = K(s,t)(1 + \lambda + \lambda^2 + \cdots)$$

$$= \frac{e^{s-t}}{1-\lambda}$$

Hence the solution is

$$g(s) = \delta(s) - \frac{\lambda}{\lambda - 1} \int_0^1 e^{s-t} f(t)\,dt$$

Volterra integral equation:

The same iterative scheme is applicable to the Volterra integral equation of the second kind. In fact, the formulas corresponding to equations (8) and (9) are, respectively

$$g(s) = f(s) + \sum_{m=1}^{\infty} \lambda^m \int_a^s K_m(s,t) f(t)\,dt$$

$$g(s) = f(s) + \lambda \int_a^s \Gamma(s,t;\lambda) f(t)\,dt$$

where the iterared kernel $K_m(s,t)$ satisfies the recurrence formula

$$K_m(s,t)=\int_t^s K(s,x)K_{m-1}(x,t)\,dx$$

with $K_1(s,t)=K(s,t)$, as before.

Example: Find the Neumann series for the solution of the integral equation

$$g(s)=(1+s)+\lambda\int_0^s (s-t)g(f)\,dt$$

Solution : We have

$$K_1(s,t)=s-t$$

$$K_2(s,t)=\int_t^s (s-x)(x-t)\,dx=\frac{(s-t)^3}{3!}$$

$$K_3(s,t)=\int_t^s \frac{(s-x)(x-t)^3}{3!}\,dx=\frac{(s-t)^5}{5!}$$

and so on. Thus,

$$g(s)=1+s+\lambda\left(\frac{s^2}{2!}+\frac{s^3}{3!}\right)+\lambda^2\left(\frac{s^4}{4!}+\frac{s^5}{5!}\right)+\cdots$$

Applications to ordinary differential equations :

The theories of ordinary differental equations are fruithful sources of integral equations. In the quest for the representation formula for the solution of a linear differential equation in such a manner so as to include the boundary conditions or initial conditions explicitly, one is always led to an integral equation. Once a boundary value or an initial value problem has been formulated in terms of an integral equation, it becomes possible to solve this problem easily.

Initial value problem: There is a fundamental relationship between Volterra integral equations and ordinary differential equations with prescribed initial values. We begin our discussion by studying the simple initial value problem.

$$y''+A(s)y'+B(s)y=F(s),\quad a\le s\le b \qquad ....(10)$$

$$y(a)=q_0,\quad y'(a)=q_1$$

where a prime implies differentiation with respect to s, and the functions A, B, and F are defined and continuous in the closed interval $a\le s\le b$.

The result of integrating the differential equation above from a to s and using the initial values above is

$$y'(s)-q_1=-A(s)y(s)-\int_a^s [B(s_1)-A'(s_1)]y(s_1)\,ds_1+\int_a^s F(s_1)\,ds_1+A(a)q_0$$

Similarly, a second integration yields

$$y(s)-q_0=-\int A(s_1)\,y(s_1)\,ds_1-\int_a^s\int_a^{s_2}[B(s_1)-A'(s_1)]y(s_1)\,ds_1\,ds_2$$

$$+\int_a^s\int_a^{s_2}F(s_1)\,ds_1\,ds_2+[A(a)q_0+q_1](s-a) \qquad ...(11)$$

With the help of the identity

$$\int_a^s\int_a^{s_n}\cdots\int_a^{s_3}\int_a^{s_2} F(s_1)\,ds_1\,ds_2\cdots ds_{n-1}\,ds_n = \frac{1}{(n-1)!}\int_a^s (s-t)^{n-1}F(t)\,dt$$

the two double integrals in equation (11) can be converted to single integrals.

Hence the relation (11) takes the form

$$y(s) = q_0 + [A(a)q_0 + q_1](s-a) + \int_a^s (s-t)F(t)\,dt$$
$$-\int_a^s [A(t) + (s-t)\{B(t) - A'(t)\}]y(t)\,dt$$

Now setting

$$K(s,t) = -\{A(t) + (s-t)[B(t) - A'(t)]\} \qquad ....(12)$$

and $$f(s) = \int_a^s (s-t)F(t)\,dt + [A(a)q_0 + q_1](s-a) + q_0$$

we have the Volterra integral equation of the second kind:

$$y(s) = f(s) + \int_a^s K(s,t)\,y(t)\,dt. \qquad ....(13)$$

Boundary value problem :

Just as initial value problems in ordinary differential equation lead to Volterra type integral equation, boundary value problems in ordinary differential equation lead to Fredholm type integral equations. Let us illustrate by the problem

$$y''(s) + A(s)y' + B(s)y = F(s), \quad a \le s \le b \qquad ....(14)$$

$$y(a) = y_0, \quad y(b) = y_1$$

When we integrate equation (14) from a to s and use the boundary condition $y(a)=y_0$, we get

$$y'(s) = C + \int_a^s F(s)\,ds - A(s)\,y(s) + A(a)y_0 + \int_a^s [A'(s) - B(s)]y(s)\,ds,$$

where C is a constant of integration.

A second integration similarly yields

$$y(s) - y_0 = [C + A(a)\,y_0](s-a) + \int_a^s (s-t)F(t)\,dt$$
$$-\int_a^s [A(t) - (s-t)\{A'(t) - B(t)\}]y(t)\,dt \qquad ....(15)$$

The Constant C can be evaluated by setting s=b in (15) and using the second boundary condition $y(b)=y_1$

$$y_1 - y_0 = [C + A(a)y_0](b-a) + \int (b-t)F(t)\,dt$$
$$-\int [A(t) - (b-t)\{A'(t) - B(t)\}]y(t)\,dt$$

or,

$$C+A(a)y_0=\frac{1}{b-a}\Big[(y_1-y_0)-\int(b-t)F(t)\,dt$$
$$-\int\{A(t)-(b-t)[A'(t)-B(t)]\}y(t)\,dt\Big] \quad ...(16)$$

From equation (15) and (16), we have the relation

$$y(s)=y_0+\int_a^s(s-t)F(t)\,dt+\frac{s-a}{b-a}\Big[(y_1-y_0)-\int(b-t)F(t)\,dt\Big]$$
$$-\int_a^s\{A(t)-(s-t)[A'(t)-B(t)]\}y(t)\,dt$$
$$+\int\left[\frac{(s-a)}{(b-a)}\right]\{A(t)-(b-t)[A'(t)-B(t)]\}y(t)\,dt$$

The above equation can be written as the Fredholm integral equation

$$y(s)=f(s)+\int K(s,t)\,y(t)\,dt$$

povided we set

$$f(s)=y_0+\int_a^s(s-t)F(t)\,dt+\left[\frac{s-a}{b-a}\Big\{y_1-y_0-\int(b-t)F(t)\,dt\Big\}\right] \quad ...(17)$$

and

$$K(s,t)=\begin{cases}\dfrac{s-a}{b-a}\{A(t)-(b-t)[A'(t)-B(t)]\}, & s<t\\ A(t)\left\{\dfrac{s-a}{b-a}-1\right\}-[A'(t)-B(t)]\left[(t-a)\dfrac{(b-s)}{(b-a)}\right], & s>t\end{cases} \quad ....(18)$$

For the special case when A and B are constants, a=0, b=1, and $y(0)=y(1)=0$, the preceding kernel simplifies to

$$K(s,t)=\begin{cases}Bs(1-t)+As, & s<t\\ Bt(1-s)+As-A, & s>t\end{cases}$$

Example : Reduce the initial value problem

$$y''(s)+\lambda y(s)=F(s), \quad 0\le s$$
$$y(0)=1, \quad y'(0)=0$$

to a Volterra integral equation.

Solution : Comparing the equation and initial conditions with (10) we have

$$A(s)=0, \ B(s)=\lambda .$$

Therefore, the relations (12) through (13) become

$$K(s,t)=\lambda(t-s)$$
$$f(s)=1+\int_0^s(s-t)F(t)\,dt$$

and $y(s) = 1 + \int_0^s (s-t)F(t)\,dt + \lambda \int_0^s (t-s)\,y(t)\,dt$

Example : Reduce the boundary value problem

$$y''(s) + \lambda P(s)y = Q(s), \quad a \le s \le b$$
$$y(a) = 0, \quad y(b) = 0$$

to a Fredholm integral equation.

Solution : Comparing the equation and boundary conditions with the notation of (14) we have A = 0, B = $\lambda P(s)$, $F(s) = Q(s)$, $y_0 = 0$, $y_1 = 0$. Substitution of these values in the rel;ation (17) and (18) yields

$$f(s) = \int_a^s (s-t)Q(t)\,dt - \frac{s-a}{b-a}\int_a^b (b-t)Q(t)\,dt$$

$$\text{and } K(s,t) = \begin{cases} \lambda P(t)\dfrac{(s-a)(b-t)}{(b-a)} & s < t \\ \lambda P(t)\dfrac{(t-a)(b-s)}{(b-a)}, & s > t. \end{cases}$$

Hence the integral equation is

$$y(s) = f(s) + \int_a^b K(s,t)\,y(t)\,dt.$$

Exercise

1. Solve the integral equations

(i) $\phi(x) = x - \int_0^x (x-t)\phi(t)\,dt$ (ii) $\phi(x) = 1 + \lambda\int_0^x xt\,\phi(t)\,dt$

(iii) $\phi(x) = f(x) + \lambda\int_0^x e^{x-y}\,\phi(y)\,dy$ (iv) $\phi(x) = f(x) + \lambda\int_0^1 e^{x-t}\,\phi(t)\,dy$

(v) $\phi(x) = x + \lambda\int_0^1 (x+s)\phi(s)\,ds$ (vi) $\phi(x) = 1 + \lambda\int_0^1 (1-3xt)\phi(t)\,dt$

(vii) $u(x) = e^x + \lambda\int_0^{10} x + u(t)\,dt$ (viii) $u(x) = e^x + \lambda\int_0^{10} xtu(t)\,dt$

(ix) $\phi(x) = \sin x + 2\int_0^x e^{x-t}\phi(t)\,dt$

2. Convert the differential equation

$$y''(t) - 3y'(t) + 2y(t) = 4\sin t, y(0) = 1, y'(0) = -2$$

into an integral equation.

QUESTION PAPERS
2006
Paper 205
(New Syllabus)
Full Marks 80

1. *(a)* Define Fredholm integral equations. Hence solve the equation

$$u(x) = f(x) + \lambda\int_a^b k(x, t)u(t)dt$$

when the kernel $k(x, t)$ is degenerate. 8

(b) Determine the characteristic values of the integral equation .

$$u(x) = f(x) + \int_0^\pi \sin(x+t)u(t)dt$$

where x and t are real variables and hence solve it. 7

Or

Find the integral equation corresponding to the differential equation

$$y'' + xy' + y = 0$$

when $y(0) = 1$, $y'(1) = 0$.

2. Distinguish between Volterra and Fredholm integral equations. What are the conditions under which Volterra integral equation can be considered as particular cases of Fredholm integral equation? Discuss the method of successive approximation of the following equation under conditions to be stated : 1+2+7=10

$$u(x) = f(x) + \int_a^x k(x, t)u(t)dt$$

3. *(a)* Define Fourier transform of a function $F(x)$. Hence derive the Fourier sine transform and cosine transform of $F(x)$ for suitable values of x. Also write down the corresponding inverse Fourier sine transform and cosine transforms. 1+3+2=6

(b) Find the Fourier transform of $F(x)$, where

$$F(x) = \begin{cases} 1 \ , & |x| < a \\ 0 \ , & |x| > a \end{cases}$$ 4

Or

Find the Fourier cosine representation of the function $f(x)$ defined by

$$f(x)=\begin{cases} x & , \quad 0<x<1 \\ 2-x & , \quad 1<x<2 \\ 0 & , \quad x>2 \end{cases}$$

4. (a) Define inverse Laplace transform $f(s)$ of a function $F(t)$. Hence, prove the change of its scale property, viz.,

$$L^{-1}\{f(as)\}=\frac{1}{a}F\left(\frac{t}{a}\right),\quad a>0$$

if

$$L^{-1}\{f(s)\}=F(t)$$ 5

(b) Evaluate : 4

$$L^{-1}\left\{\frac{s+1}{(s-1)^2(s+2)}\right\} \quad or \quad L\{t^2e^{-t}\sin 3t\}$$

(c) By using the method of Laplace transform, solve the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x=0$$

where

$$x(0)=0,\quad x'(0)=1$$ 6

Or

Using the method of Laplace transform, solve the partial differential equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\qquad x>0,\ t>0$$

and $u(0, t)=1,\qquad u(x, 0)=0$

5. (a) What do you mean by functionals, admissible functions, zeroth-order proximity and first-order proximity? In the problems of extremals, what is the necessity of first-order proximity? 7

(b) Determine the Euler's differential equation for the extremal of the functional

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y', y'')dx$$

where F is the functions of the arguments x, y, y', y'' and the admissible function and its derivatives are continuous which satisfy prescribed conditions at the boundaries x_1 and x_2. 8

Or

Show that the shortest distance between two points on the surface of a sphere is the arc of a great circle.

6. (a) Find the extremal which makes the integral

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')dx$$

stationary with the prescribed boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ and is subject to the subsidiary condition

$$\int_{x_1}^{x_2} g(x, y, y')dx = C, \text{ a constant}$$ 8

(b) Find the extremal for the extremum of

$$I[y, z] = \int_0^1 (y'^2 + z'^2 - 4xz' - 4z)dx$$

subject to the constraint

$$\int_0^1 (y'^2 - xy' - z'^2)dx = 2$$

where

$$y(0) = 0, \; z(0) = 0 \text{ and } y(1) = 1, \; z(1) = 1$$ 7

Or

If a particle of mass m is constrained to move on a given surface $G(x, y, z) = 0$ and if no forces act on it, then show that it slides along a geodesic (curve of minimum length).

* * *

2008

MATHEMATICS

Paper : 205

(Mathematical Methods)

(New Syllabus)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* Distinguish between Volterra and Fredholm integral equations. Classify the following integral equation and hence solve it : 1+1+6=8

$$\phi(x) = x + \lambda \int_0^1 (xy^2 + x^2 y)\, \phi(y)\, dy$$

(b) Determine the characteristic values of the integral equation

$$u(x) = f(x) + \int_0^{\pi} \sin(x+t)\, u(t)\, dt$$

where x and t are real variables, and hence solve it. 7

Or

(i) For what value of λ, the function $\phi(x) = 1 + \lambda x$ is a solution of the integral equation

$$x = \int_0^x e^{x-t}\, \phi(t)\, dt \, ?$$ 3

(ii) Prove that the integral equation

$$\phi(x) = \frac{1}{e^2 - 1} \int_0^1 2e^x \cdot e^t \phi(t)\, dt$$

does not have characteristic number and characteristic function. 4

2. Write a note about the utility of converting a differential equation into an integral equation. Form an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = 0$.

3+7=10

3. (a) Using Fourier integral representation, show that

$$\int_0^\infty \frac{\cos\left(\lambda \frac{\pi}{2}\right)\cos(\lambda x)}{1-\lambda^2} d\lambda = \begin{cases} \frac{\pi}{2}\cos x, & |x| < \frac{\pi}{2} \\ 0 \,, & |x| > \frac{\pi}{2} \end{cases}$$ 5

(b) Define Fourier transform of a function $F(x)$. Hence, derive the Fourier sine and cosine transforms of $F(x)$ for suitable values of x. Also write down the corresponding inverse Fourier sine transform and cosine transform. 5

4. (a) Find the Laplace transform of

$$(te^{at} \sin at)$$ 4

(b) Find the value of

$$L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$$ 4

(c) Using the method of Laplace transform, solve the differential equation

$$\frac{d^2x}{dt^2} + n^2 x = a \sin(nt + \alpha)$$

where $x(0) = \dot{x}(0) = 0$ at $t = 0$. 7

Or

Using the method of Laplace transform, solve the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 20 \sin 2t$$

given that $x(0) = -1$, $x(0) = 2$ at $t = 0$.

5. (a) Define admissible function for a functional in a stationary value problem. Find the differential equation for extremum value of

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y', y'')\, dx$$

with the prescribed boundary conditions (to be stated) where

$$y' = \frac{dy}{dx} \text{ and } y'' = \frac{d^2y}{dx^2}$$ 2+6=8

(b) Show that the shortest distance between any two points on the surface of a sphere is the arc of a great circle. 7

6. (a) What do you mean by 'isoperimetric' problem? Hence, maximize the functional

$$I[y(x)] = \int_0^1 y\,dx$$

subject to the side condition

$$\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = L \text{ (constant)}$$

and the boundary conditions $y(0) = 0$ and $y(1) = 0$. 1+7=8

(b) Determine the number of functions $y(x), z(x), \cdots$ of x such that the integral

$$I[y(x), z(x), \cdots] = \int_{x_1}^{x_2} F(x, y, z, \cdots, y', z', \cdots)\,dx$$

is an extremum with the assigned boundary conditions at the end points. 7

Or

Find the curve joining two given points $A(0, 0)$ and $B(x_1, y_1)$ so that a particle moving along this curve starting from A reaches B in the shortest time, friction and resistance of the medium being neglected.

***

2009

MATHEMATICS

Paper : 205

(Mathematical Methods)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer any *three* parts : 5×3=15

(a) Obtain Fredholm integral equation of 2nd kind corresponding to the boundary value problem

$$\frac{d^2\phi}{dx^2}+\lambda\phi=x\,;\ \phi(0)=0,\ \phi'(1)=0$$

(b) Solve the integral equation

$$\phi(x)=f(x)+\lambda\int_{-1}^{1}(xt+x^2t^2)\phi(t)\,dt$$

Also find its resolvent kernel.

(c) Solve the following integral equation by the method of successive approximation :

$$\phi(x)=\frac{5x}{6}+\frac{1}{2}\int_0^1 xt\,\phi(t)\,dt$$

(d) Show that the homogeneous Fredholm integral equation

$$\phi(x)=\lambda\int_0^1(3x-2)\phi(t)\,dt$$

has no eigenvalues and eigenfunctions.

2. Answer any *two* parts : 5×2=10

(a) Show that the resolvent kernel of the integral equation

$$\phi(x)=1+\lambda\int_0^x xt\,\phi(t)\,dt$$

is $xte^{\lambda/3}(x^3-t^3)$ and hence solve the equation.

(b) Using the method of successive approximation, solve the integral equation

$$\phi(x) = 1 + \int_0^x \phi(t)\,dt$$

taking the initial approximation as $\phi_0(x) = 0$.

(c) With the aid of resolvent kernel, find the solution of the following integral equation :

$$\phi(x) = \sin x + 2\int_0^x e^{x-t}\,\phi(t)\,dt$$

3. Answer any *two* parts : 5×2=10

(a) Find the Fourier transform of

$$f(x) = 1,\ |x| < a$$
$$= 0,\ |x| > a$$

and hence evaluate

(i) $\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s}\,ds$

(ii) $\int_0^{\infty} \frac{\sin s}{s}\,ds$

(b) Define Fourier integral theorem and hence show that the following two forms are equivalent

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha$$

and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha \int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\,du$$

(c) Show that

$$\int_0^{\infty} \frac{(x\cos x - \sin x)^2}{x^6}\,dx = \frac{\pi}{15}$$

4. (a) Answer any *three* parts : 3×3=9

(i) Prove that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$ and hence find $L\left\{\frac{\sin at}{t}\right\}$.

(ii) Find the function $f(t)$ whose Laplace transform is $\log\left(1+\frac{\omega^2}{s^2}\right)$.

(iii) Evaluate by using convolution theorem $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

(iv) Show that $\int_0^\infty t^3 e^{-t}\sin t\,dt = 0$.

(b) Using the method of Laplace transform, solve the differential equation

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=4e^{2t}$$

given that $y(0)=3$, $y'(0)=5$. 6

Or

Using the method of Laplace transform, find the solution of

$$\frac{\partial U}{\partial x}=2\frac{\partial U}{\partial t}+U,\ U(x,0)=6e^{-3x}$$

which is bounded for $x>0$, $t>0$.

5. *(a)* Show that a necessary condition for the functional

$$I[y(x)]=\int_a^b f(x, y, y')\,dx$$

to be an extremum is that

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$$

subject to the boundary conditions $y(a)=y_1$ and $y(b)=y_2$ where y_1 and y_2 are prescribed at the fixed boundary points a and b. 7

(b) Answer any *two* parts : 4×2=8

(i) Find the curves on which the functional

$$\int_0^1[(y')^2+12xy]\,dx$$

with $y(0)=0$, $y(1)=1$ can be extremized.

(ii) Test for an extremum the functional

$$I[y(x)]=\int_0^1(xy+y^2-2y^2y')\,dx$$

with $y(0)=1$, $y(1)=2$.

(iii) Find the curve with fixed boundary points (x_1, y_1) and (x_2, y_2) such that its rotation about the axis of abscissa gives rise to a surface of revolution of minimum surface area.

6. (a) Find the extremal which makes the integral

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

stationary with the prescribed boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, subject to the subsidiary condition

$$\int_{x_1}^{x_2} g(x, y, y')\, dx = C$$

where C is a constant. 7

(b) Answer any *two* parts : 4×2=8

(i) Find the extremals of $\int_0^{\pi} (y')^2 dx$ by boundary conditions $y(0) = 0$, $y(\pi) = 0$ with the constraint $\int_0^{\pi} y^2 dx$.

(ii) Show that a closed curve given by

$$\int_{t_1}^{t_2} (x\dot{y} - y\dot{x})\, dt$$

is a functional which can be put in the form

$$I[x(t), y(t)] = \int_{t_1}^{t_2} \Phi(x, y, \dot{x}, \dot{y})\, dt$$

where Φ is a homogeneous function of degree one in $\dot{x}$ and $\dot{y}$.

(iii) Determine the curve of length l which passes through the points $(-1, 0)$ and $(1, 0)$, and for which the area between the curve and x-axis is a maximum.

★★★

2010

MATHEMATICS

Paper : 205

(Mathematical Methods)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer any *three* parts : 5×3=15

(a) Show that the function $\phi(x) = e^x[2x - (2/3)]$ is a solution of the Fredholm integral equation

$$\phi(x) + \lambda\int_0^1 e^{x-t}\phi(t)\,dt = 2xe^x$$

when $\lambda = 2$.

(b) Using the method of successive approximation, solve the integral equation

$$\phi(x) = 2x + \lambda\int_0^1 (x+t)\,\phi(t)\,dt$$

with $\phi_0(x) = 1$.

(c) Show that the integral equation

$$\phi(x) = \lambda\int_0^1 \{\sqrt{x}\,t - \sqrt{t}\,x\}\,\phi(t)\,dt$$

does not have real characteristic numbers and characteristic functions.

(d) Solve the integral equation :

$$\phi(x) = 2x - \pi + 4\int_0^{\pi/2} \sin^2 x\,\phi(t)\,dt$$

2. Answer any *two* parts : 5×2=10

(a) Convert the differential equation

$$\frac{d^2y}{dx^2} - \sin x\frac{dy}{dx} + e^x y = x$$

with the initial conditions $y(0) = 1$, $y'(0) = -1$ into Volterra integral equation of second kind.

(b) Use the method of successive approximations to solve the integral equation

$$\phi(x) = (1+x) + \int_0^x (x-t)\,\phi(t)\,dt$$

(c) With the aid of resolvent kernel, show that the solution of the integral equation

$$\phi(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2}\,\phi(t)\,dt$$

is given by

$$\phi(x) = e^x(1+x^2)$$

3. Answer any *two* parts : 5×2=10

(a) Find the Fourier transform of

$$f(x) = 1 - x^2, \quad |x| \le 1$$
$$= 0, \quad |x| > 1$$

and hence evaluate

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\,dx$$

(b) Find the Fourier cosine and sine transforms of e^{-bx} and hence evaluate the integrals :

(i) $\int_0^\infty \frac{\cos\lambda x}{\lambda^2 + b^2}\,d\lambda$

(ii) $\int_0^\infty \frac{\lambda \sin\lambda x}{\lambda^2 + b^2}\,d\lambda$

(c) Using the Fourier sine transform, solve the partial differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad x > 0, \; t > 0$$

subject to the conditions

$$U(0, t) = 0, \; U(x, 0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$$

$U(x, t)$ is bounded.

4. *(a)* Answer any *three* parts : 3×3=9

(i) Show that

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \log \frac{b}{a}$$

(ii) Evaluate :

$$\int_0^\infty t e^{-3t} \sin t\, dt$$

(iii) Find :

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$$

(iv) Show that

$$L^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \frac{1}{2}\cos 2t$$

(b) Using the method of Laplace transform, solve the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

given that $y(0) = 2$ and $y'(0) = -4$. 6

Or

If the function $U(x, t)$ is defined for $a \le x \le b$, $t > 0$, show that

(i) $L\left\{\frac{\partial U}{\partial t}\right\} = su(x, s) - U(x, 0)$

(ii) $L\left\{\frac{\partial^2 U}{\partial t^2}\right\} = s^2 u(x, s) - sU(x, 0) - U_t(x, 0)$

where $u(x, s) = L\{U(x, t)\}$.

5. *(a)* Define admissible function for a functional in a stationary value problem. Find the differential equation for extremum value of

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y', y'')\, dx$$

under prescribed boundary conditions at the boundaries x_1 and x_2 where prime denotes differentiation with respect to x. 8

(b) Answer any *one* part : 7

(i) Find the extremal of the functional

$$\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\, dx$$

with $y(0) = 0$, $y(\pi/2) = 0$.

(ii) Show that the general solution of the Euler's equation for the integral

$$\int_a^b \frac{1}{y} \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

is $(x-h)^2 + y^2 = k^2$

6. (a) What do you mean by isoperimetric problem? Find the extremals of the isoperimetric problem

$$I[y(x)] = \int_{x_0}^{x_1} y'^2\, dx$$

given that

$$\int_{x_0}^{x_1} y\, dx = C,$$

a constant 8

(b) Find the extremal of the functional

$$I = \int_0^{\pi} (y'^2 - y^2)\, dx$$

under the conditions $y(0) = 0$, $y(\pi) = 1$ and subject to the constraint

$$\int_0^{\pi} y\, dx = 1$$

7

Or

Determine the space curve $x = x(t)$, $y = y(t)$ and $z = z(t)$ to give stationary value of the integral in the form

$$\int_{t_1}^{t_2} f(\dot{x}, \dot{y}, \dot{z})\, dt$$

lying on the surface $G(x, y, z) = 0$.